

Lecture 8

Fairness in the Stable Matching Problem

Reminder about starting recording

Stable Matching Problem

$w_1 > w_2 > w_3$



$m_3 > m_2 > m_1$

$w_2 > w_1 > w_3$



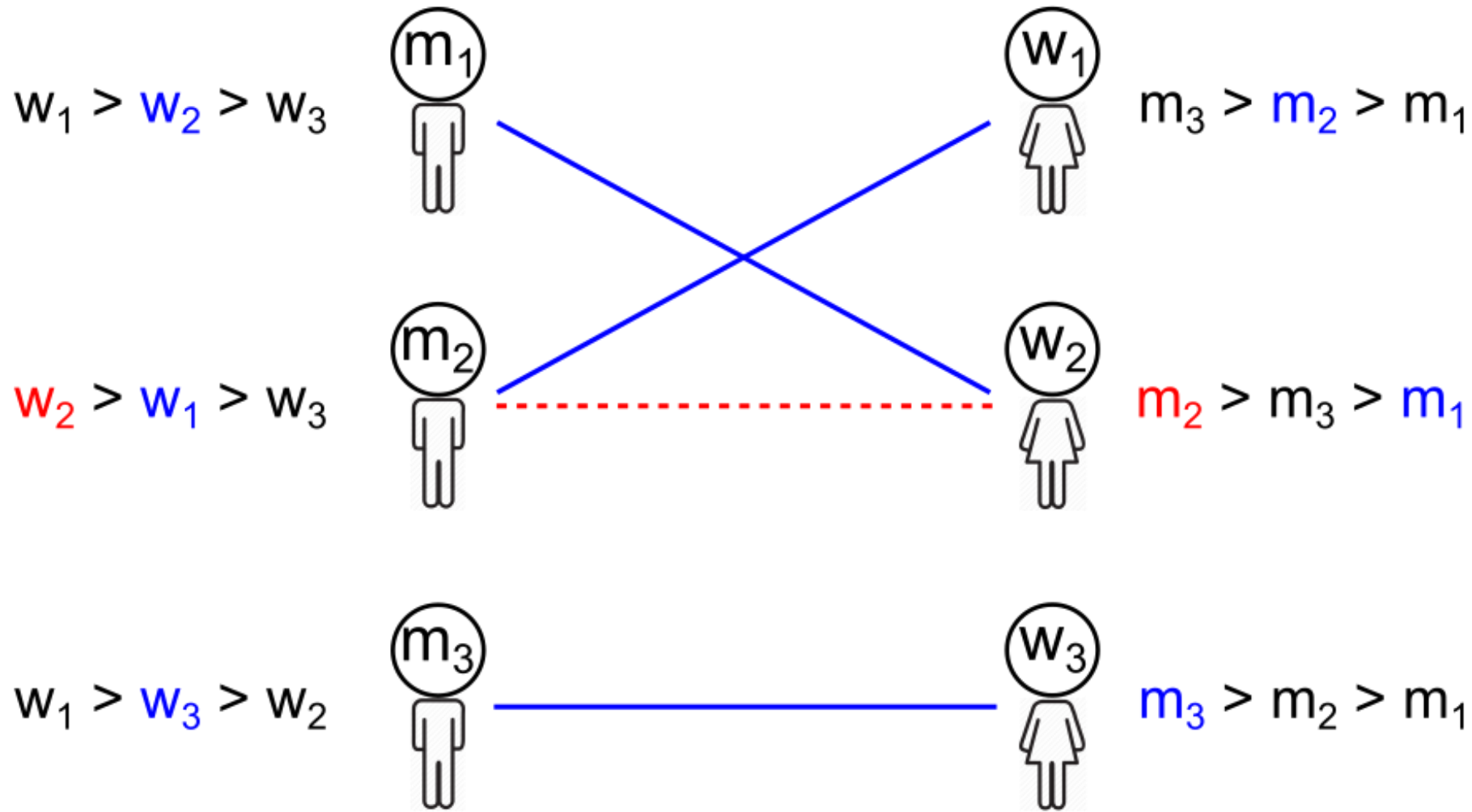
$m_2 > m_3 > m_1$

$w_1 > w_3 > w_2$



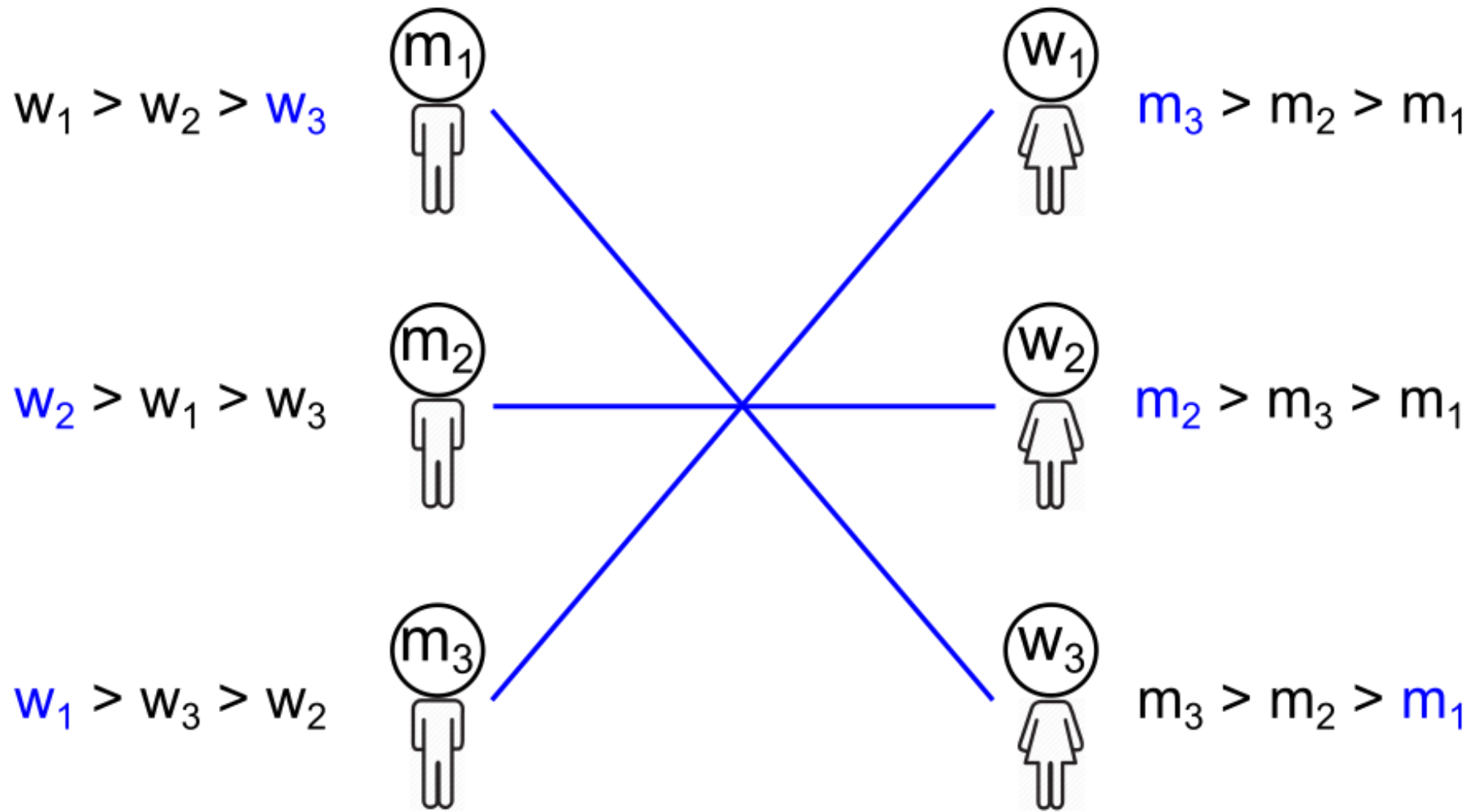
$m_3 > m_2 > m_1$

Stable Matching Problem



A matching is **stable** if there is no **blocking pair**.

Stable Matching Problem



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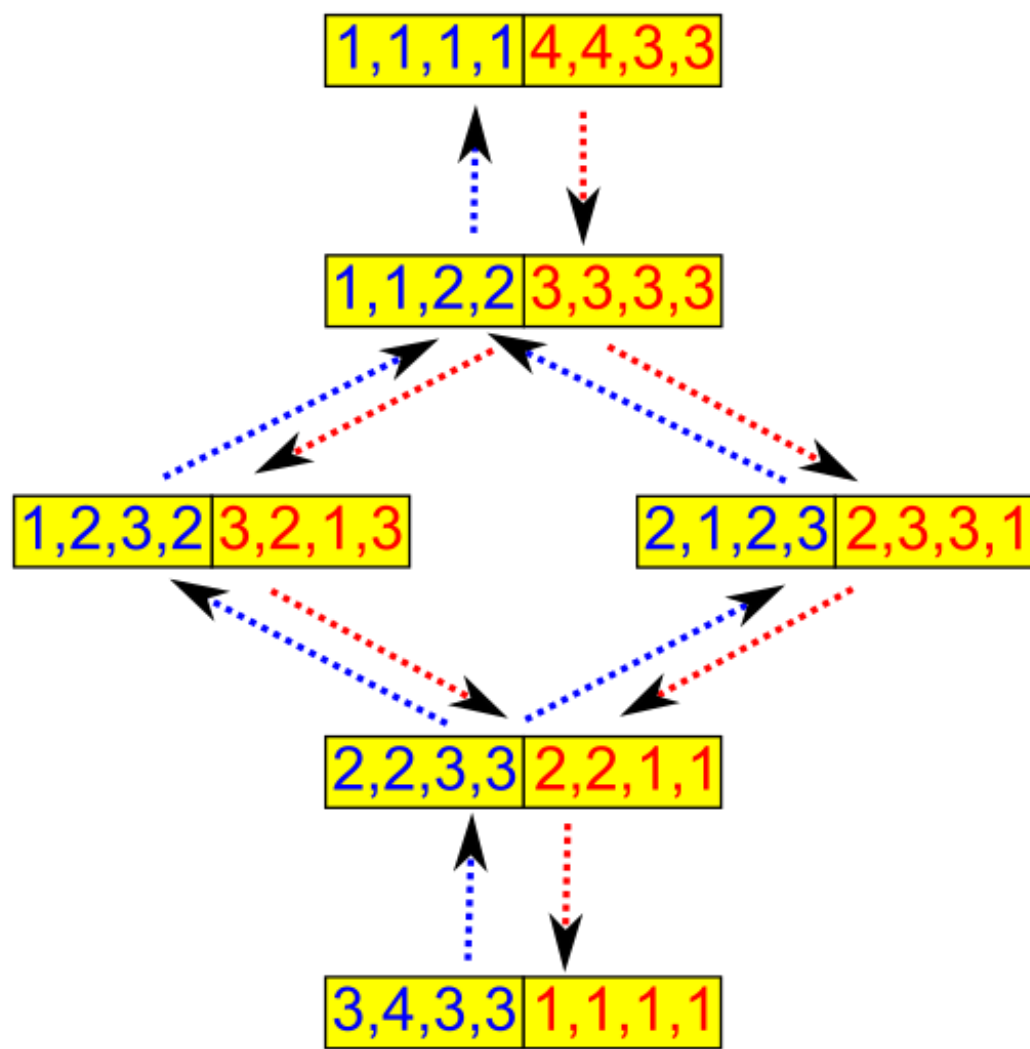
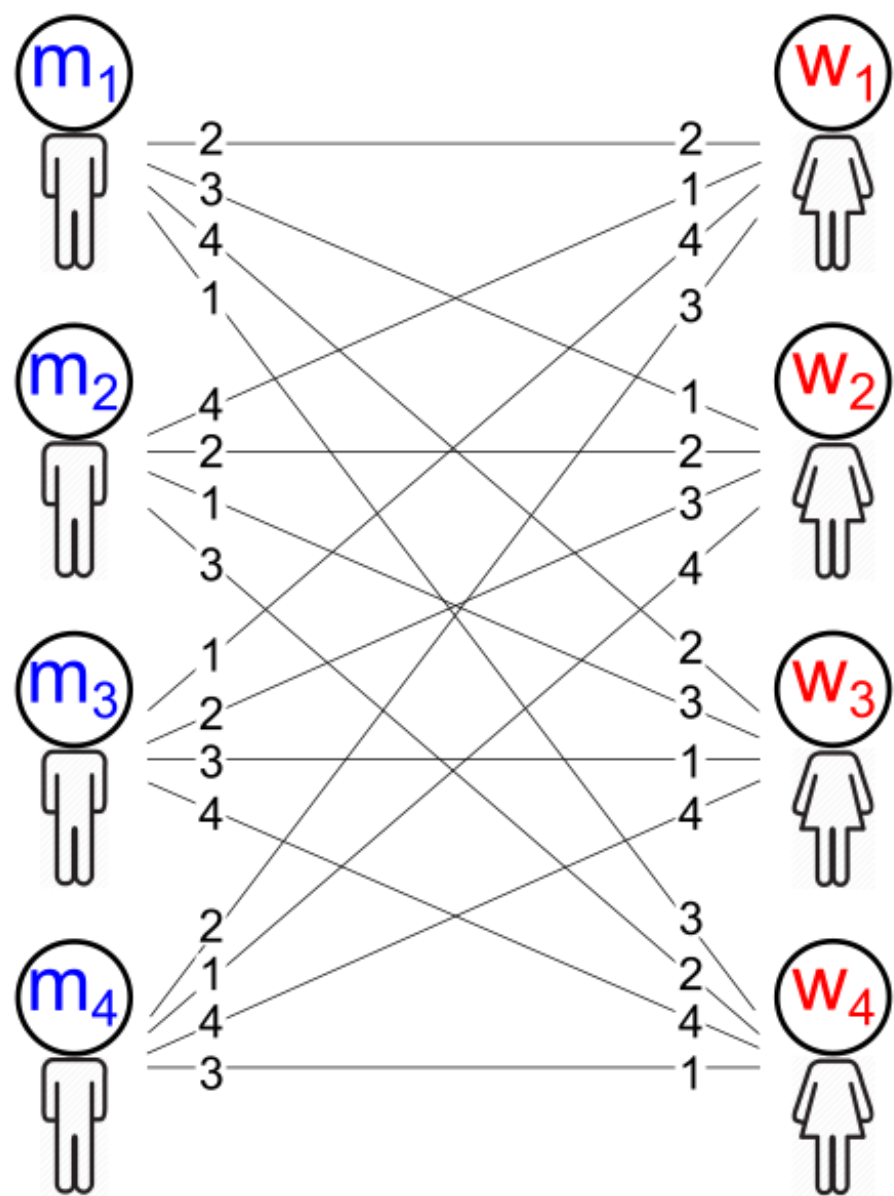
COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

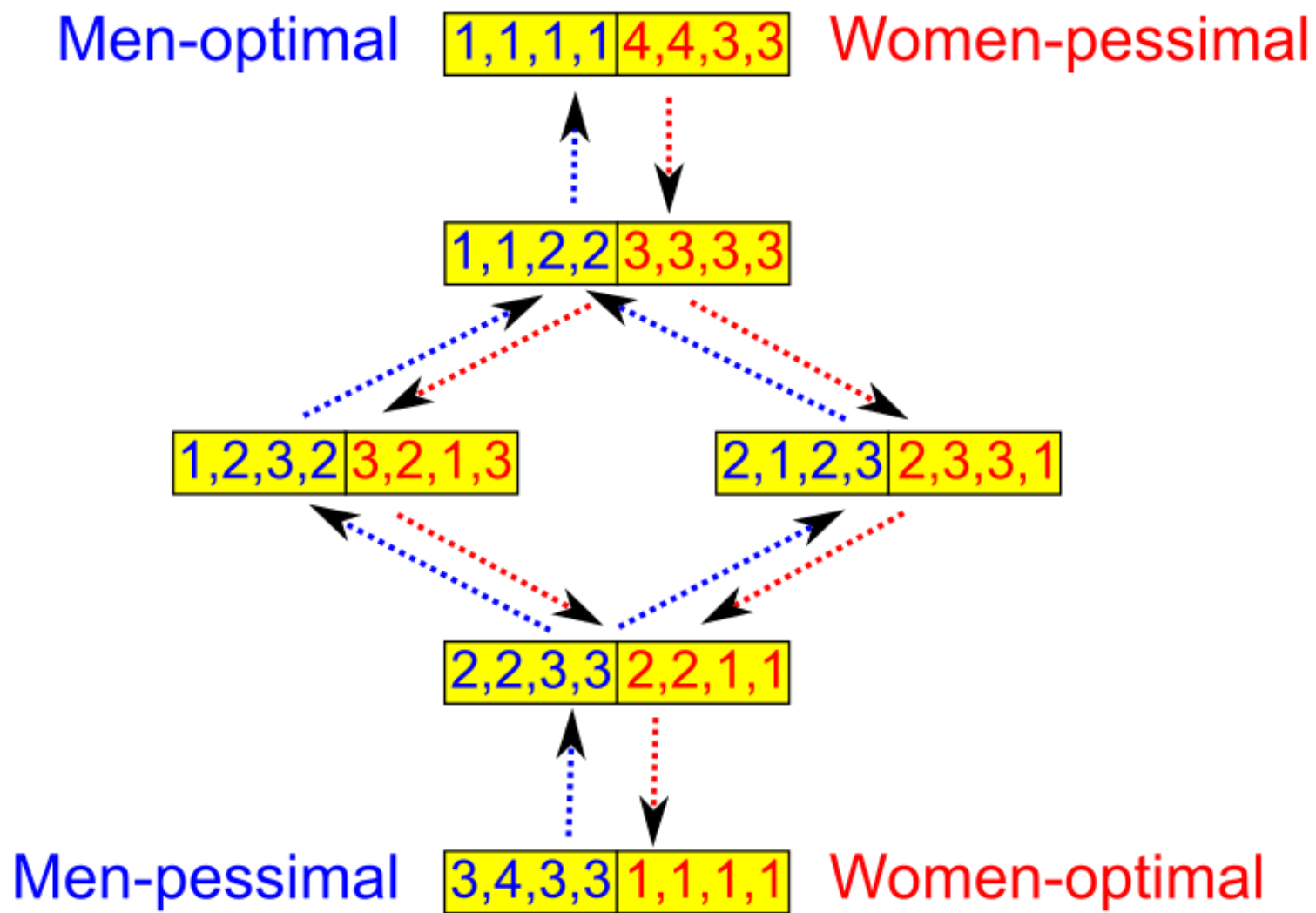
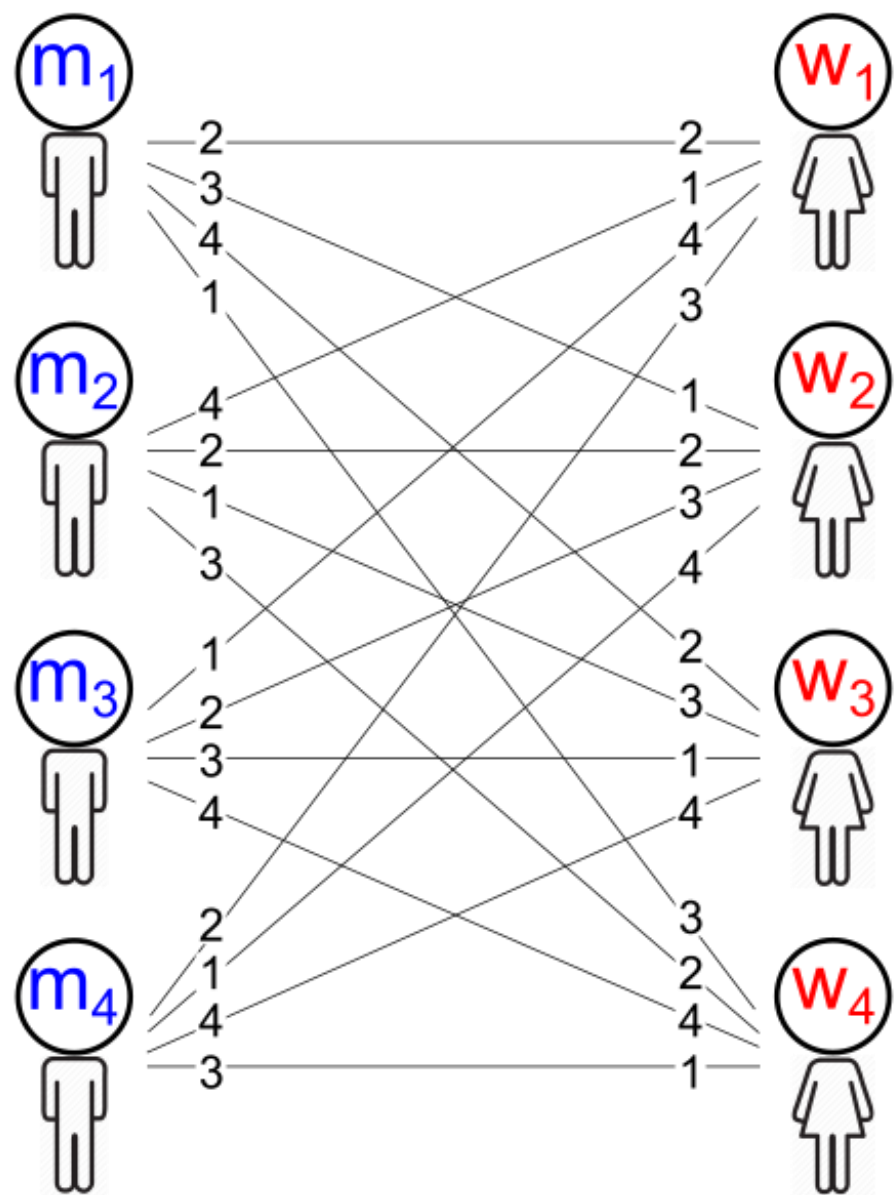
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

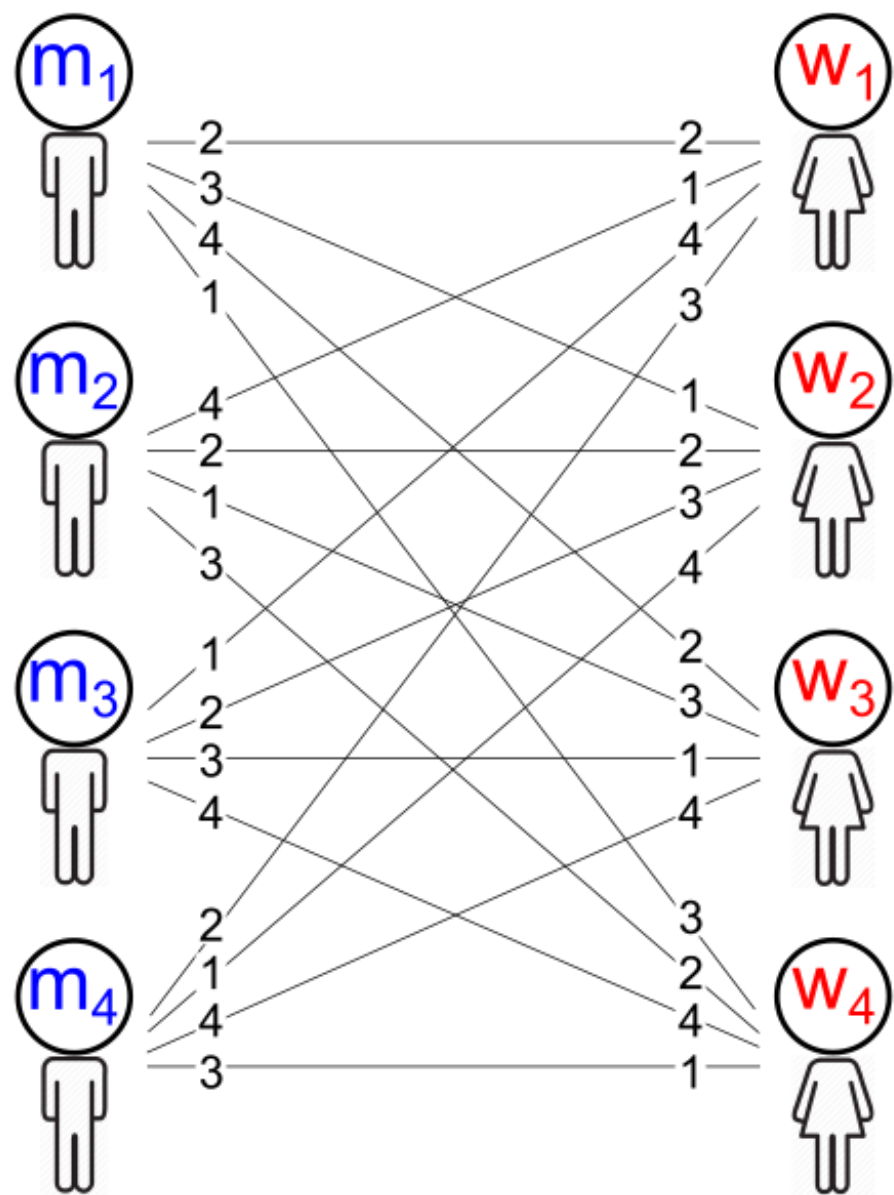


Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9-15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.

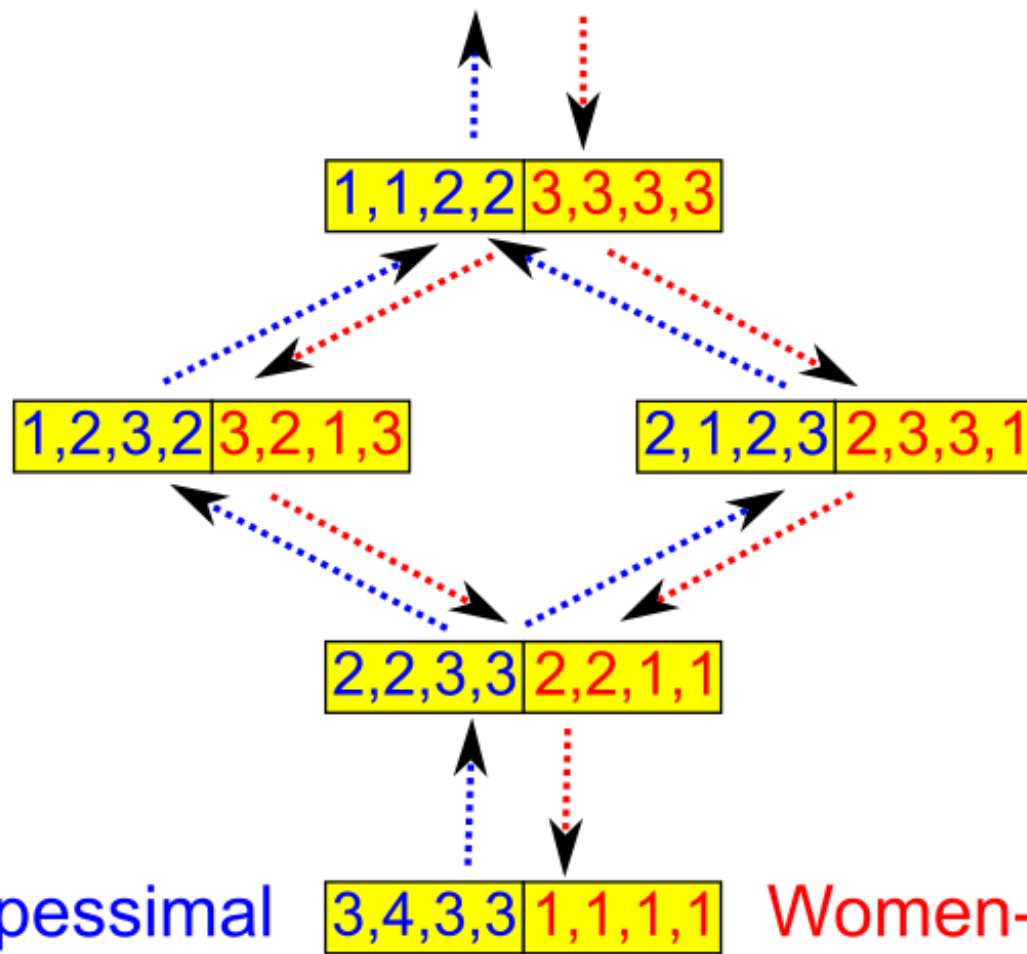




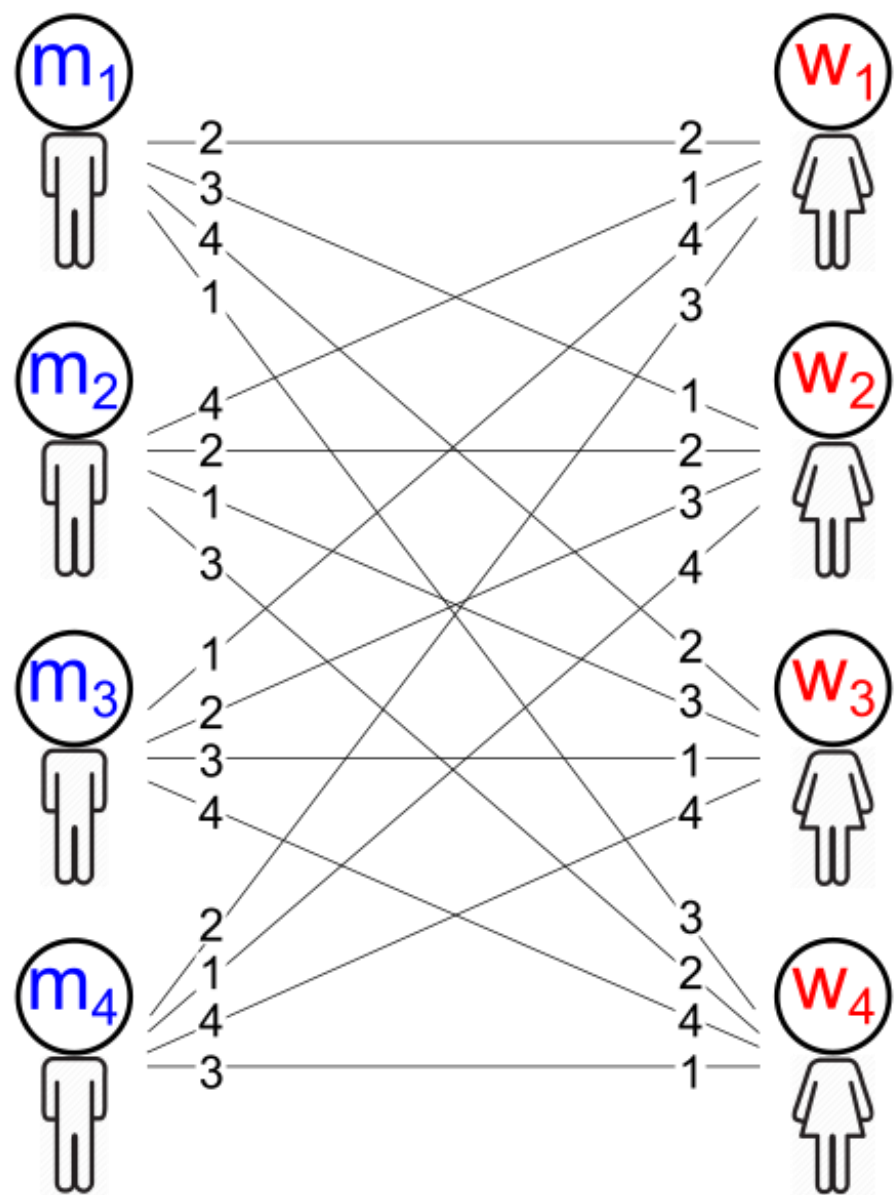


Men-proposing DA algorithm computes this

Men-optimal $1, 1, 1, 1 \mid 4, 4, 3, 3$ Women-pessimal

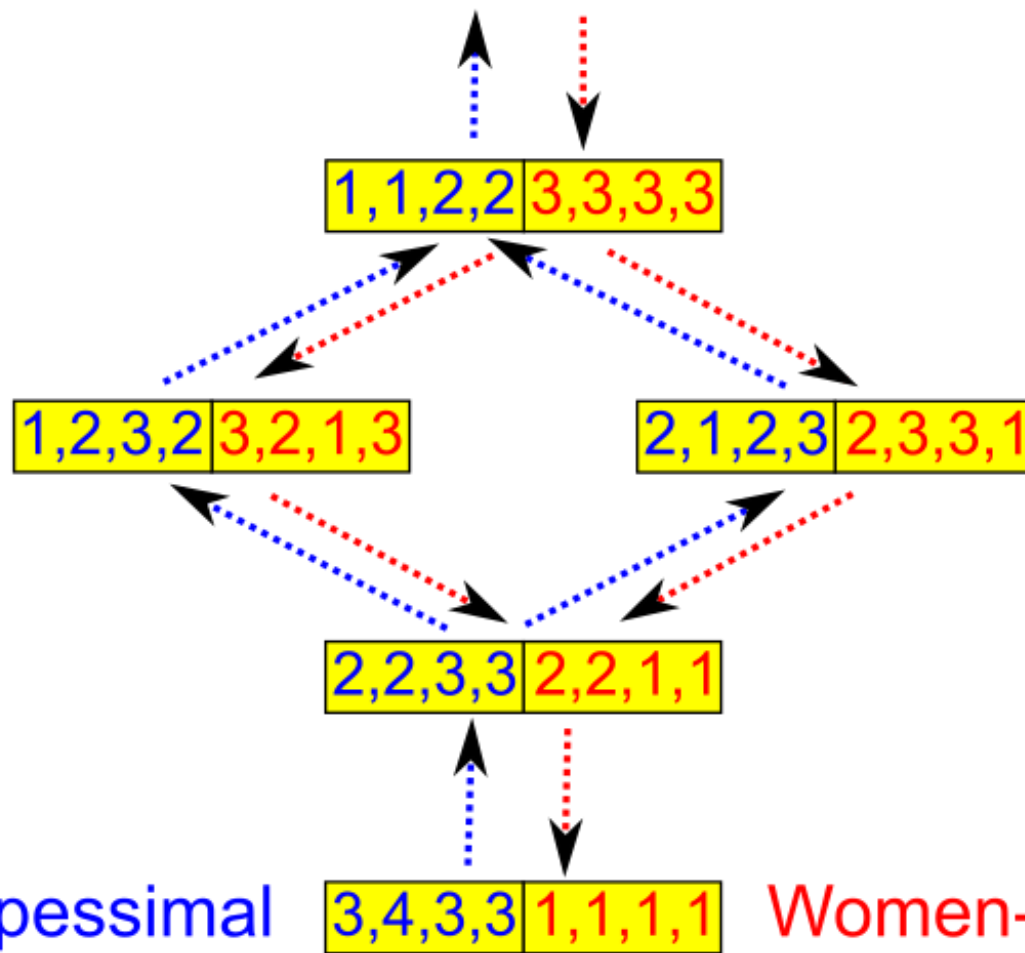


Men-pessimal $3, 4, 3, 3 \mid 1, 1, 1, 1$ Women-optimal



Men-proposing DA algorithm computes this

Men-optimal $1, 1, 1, 1$ | $4, 4, 3, 3$ Women-pessimal

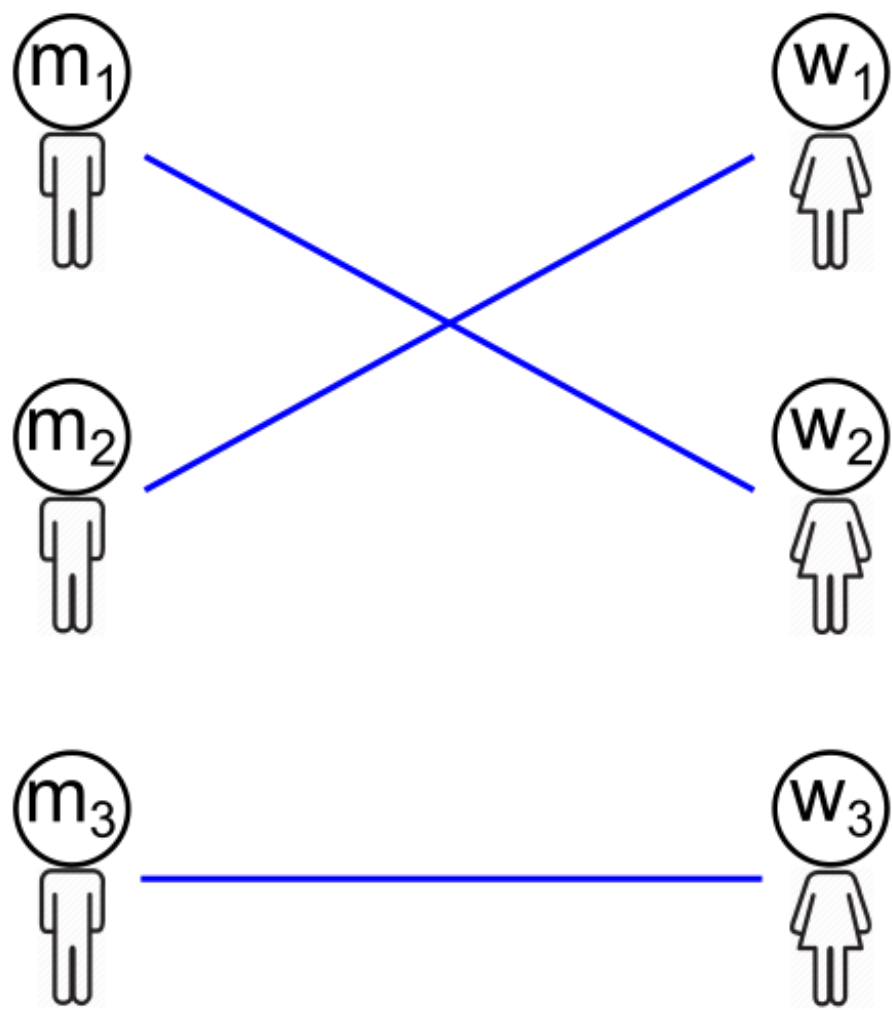


Women-proposing DA algorithm computes this

Goal for Today

Understanding the structure of the set of stable matchings through linear programming.

(This will guide us towards fair stable matchings.)



$$P = \begin{matrix} & w_1 & w_2 & w_3 \\ m_1 & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ m_2 & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ m_3 & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Fractional Stable Matching

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Any non-negative $n \times n$ matrix X satisfying the following:

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$$X_{i,j} \geq 0 \text{ for all } i \in [n] \text{ and } j \in [n]$$

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$$\sum_i X_{i,j} = 1 \text{ for all } j \in [n] \quad \textit{Every woman is fully matched.}$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \text{ for all } i \in [n] \text{ and } j \in [n] \quad \textit{Stability}$$

Fractional Stable Matching

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Any integral stable matching is also a fractional stable matching.

Fractional Stable Matching

Any non-negative $n \times n$ matrix X satisfying the following:

Any convex combination of integral stable matchings is also a fractional stable matching.

$$X = \sum_k \lambda_k P^k$$

integral stable matching

such that $\lambda_k \geq 0$ for all k and $\sum_k \lambda_k = 1$

Any integral stable matching is also a fractional stable matching.

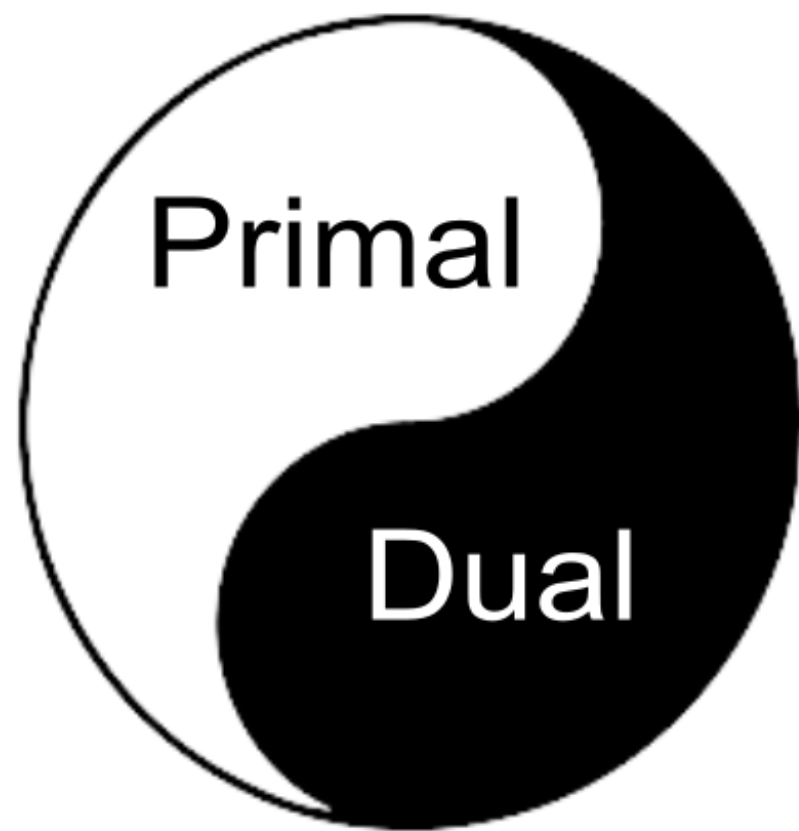
[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Coming up...

An elegant geometric proof that uses LP duality and its application in fair stable matchings.



Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Primal

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$$X_{i,j} \geq 0 \quad \forall i, j$$

Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Primal

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Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i \quad \alpha_i$$

$$\sum_i X_{i,j} = 1 \quad \forall j \quad \beta_j$$

$\gamma_{i,j}$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

Don't worry about us.

Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i \quad \alpha_i$$

Combine the constraints in order to construct an upper bound on the objective.

$$-X_{i,j} - \sum_{k:w_k \succ m_i} X_{i,k} - \sum_{k:m_k \succ w_j} X_{k,j} \leq -1 \quad \forall i, j \quad \gamma_{i,j}$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Don't worry about us.

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Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k:w_k > m_i w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k > w_j m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

Primal

Let's combine these.

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k:w_k > m_i w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k > w_j m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

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$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j > m_i w_k} \gamma_{i,k} - \sum_{k: m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

subject to this
being at least 1

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

as long as
 $\gamma_{i,j} \geq 0 \forall i, j$

subject to this
being at least 1

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

By Gale and Shapley's result,
primal is always feasible!

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

Let $X_{i,j}$ be a feasible primal solution.

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

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Dual

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$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_i X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

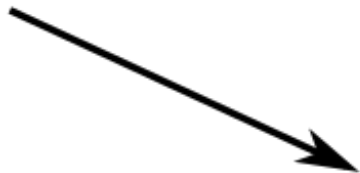
$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint
from primal



$$\sum_{i,j} X_{i,j}$$

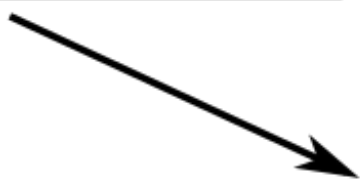
$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint
from primal

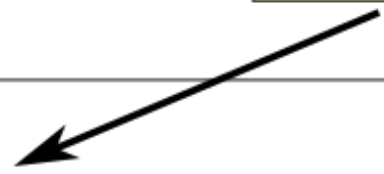


$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Dual feasible!



Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint
from primal

Objective is equal
to that in the primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint
from primal

Objective is equal
to that in the primal

$$\sum_{i,j} X_{i,j}$$

By strong duality:

$X_{i,j}$ must be *primal optimal*, and $X_{k,j} + X_{i,j} \geq 1 \quad \forall i,j$

$\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ must be *dual optimal*.

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i,j

Dual

Stability constraint
from primal

Objective is equal
to that in the primal

By complementary slackness:

For any primal feasible X ,

$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k: w_k > m_i w_j} X_{i,k} + \sum_{k: m_k > w_j m_i} X_{k,j} = 1.$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Proof by picture (and LP duality)

[Teo and Sethuraman, *MOR* 1998].

[Vande Vate, *Oper. Res. Let.* 1989]

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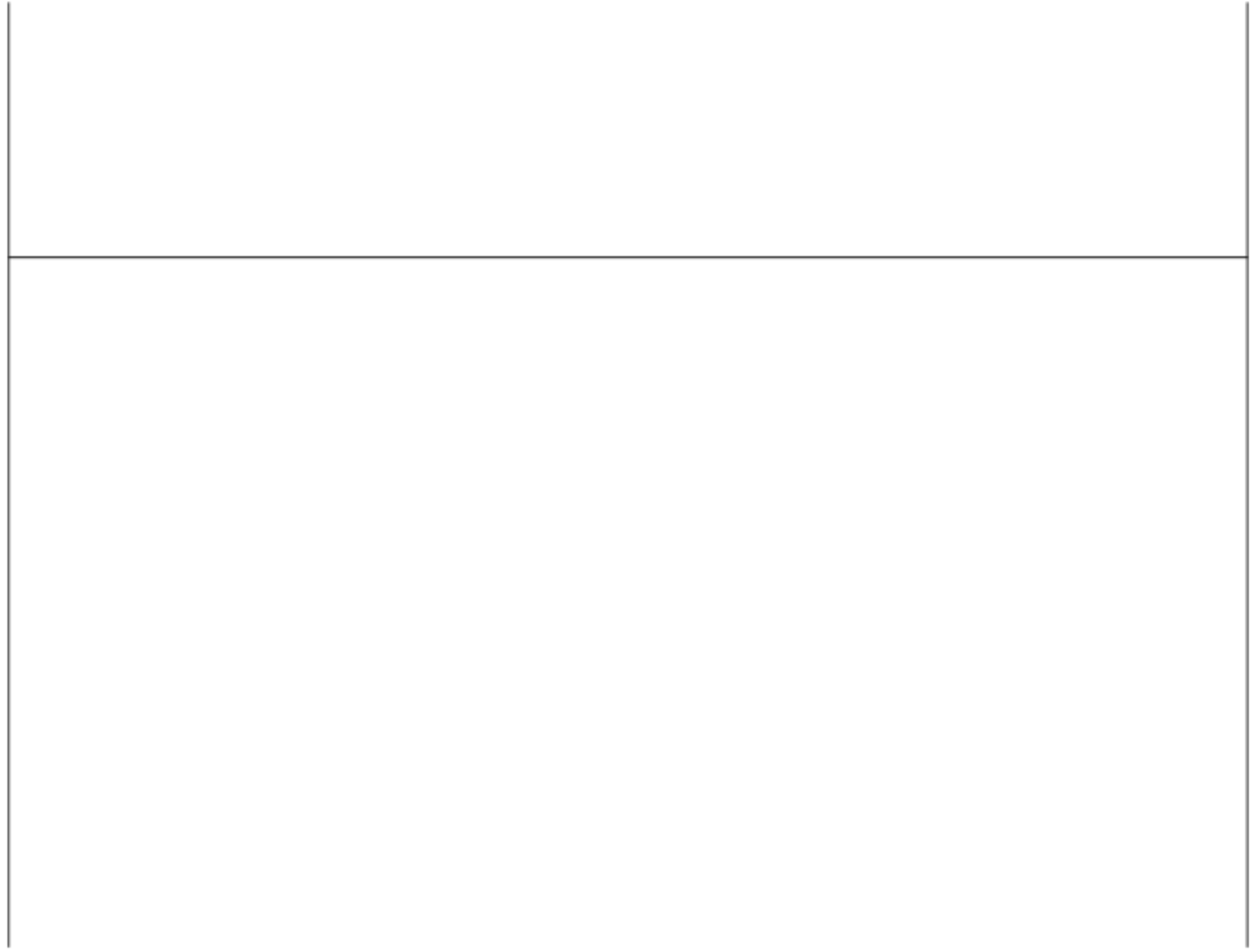
Recall complementary slackness:

For any primal feasible X ,

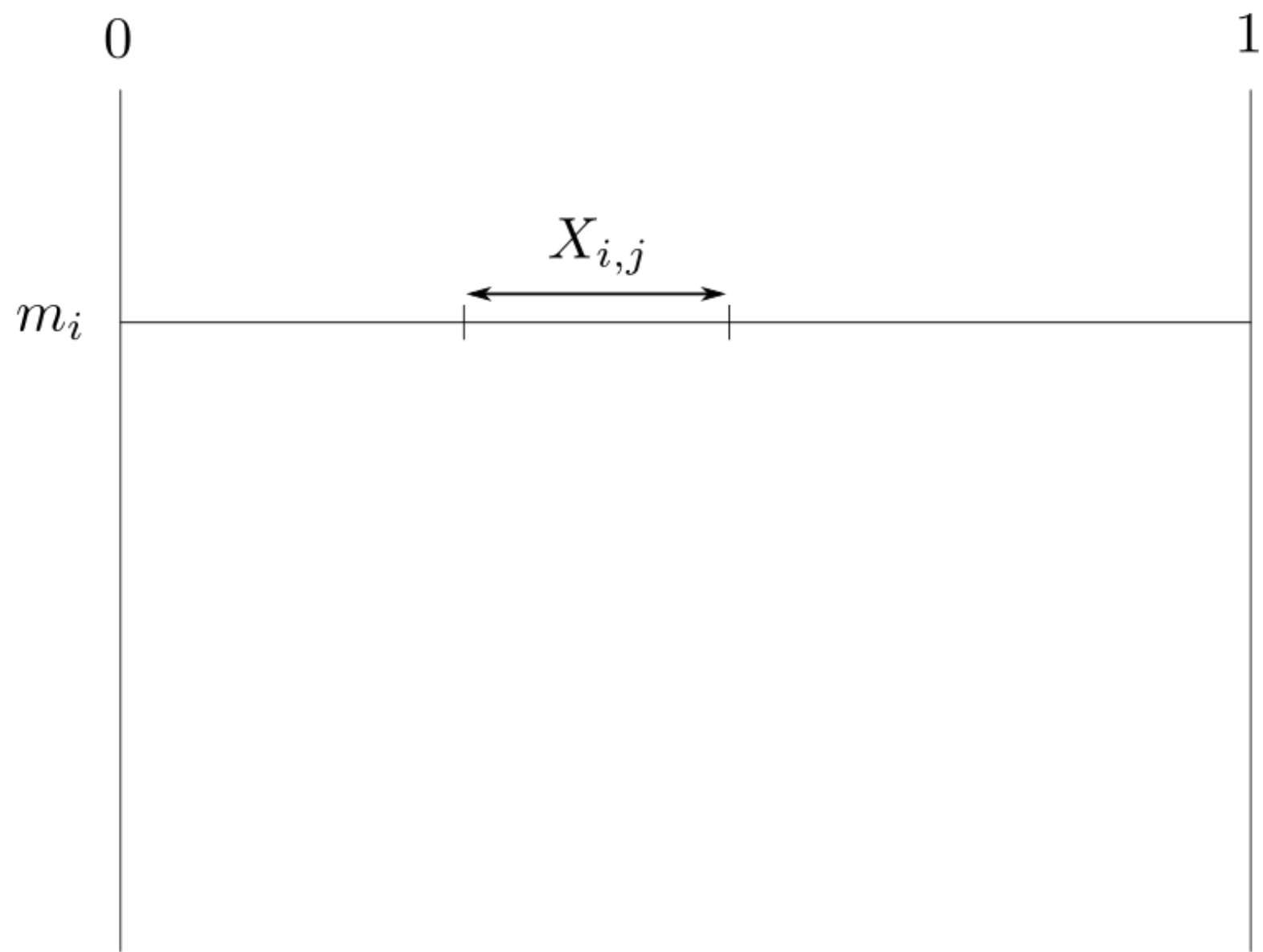
$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} = 1.$$

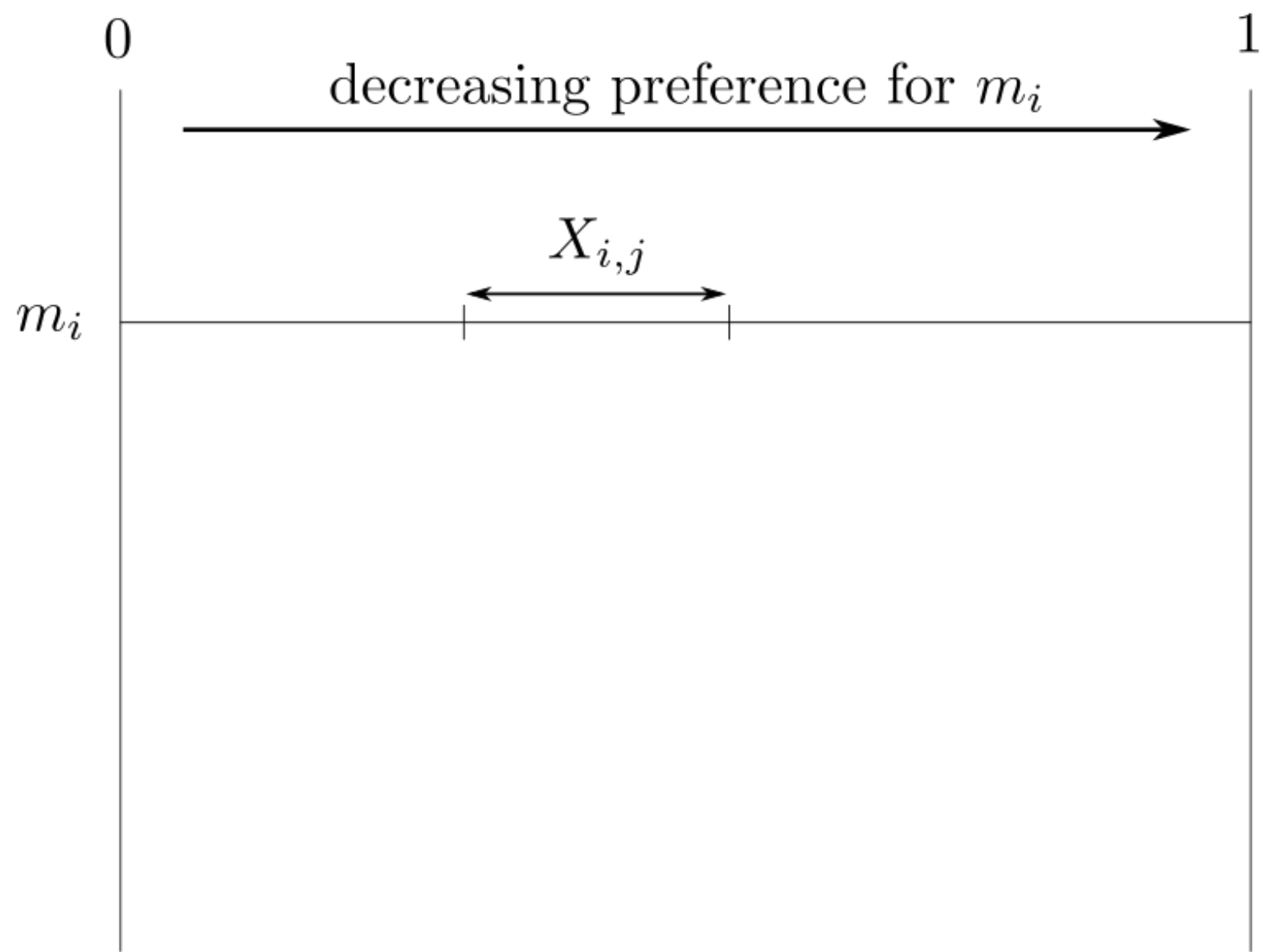
0

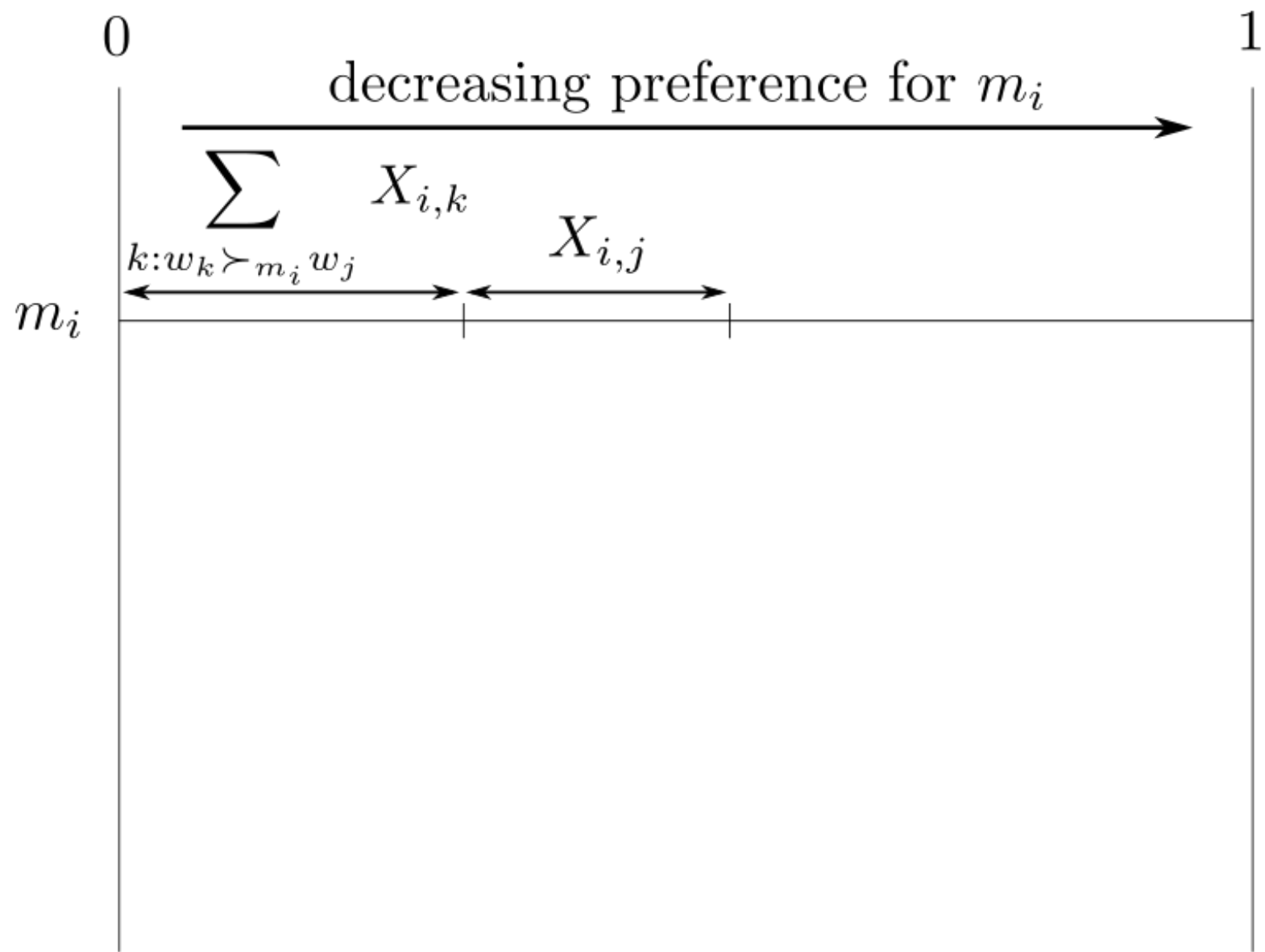
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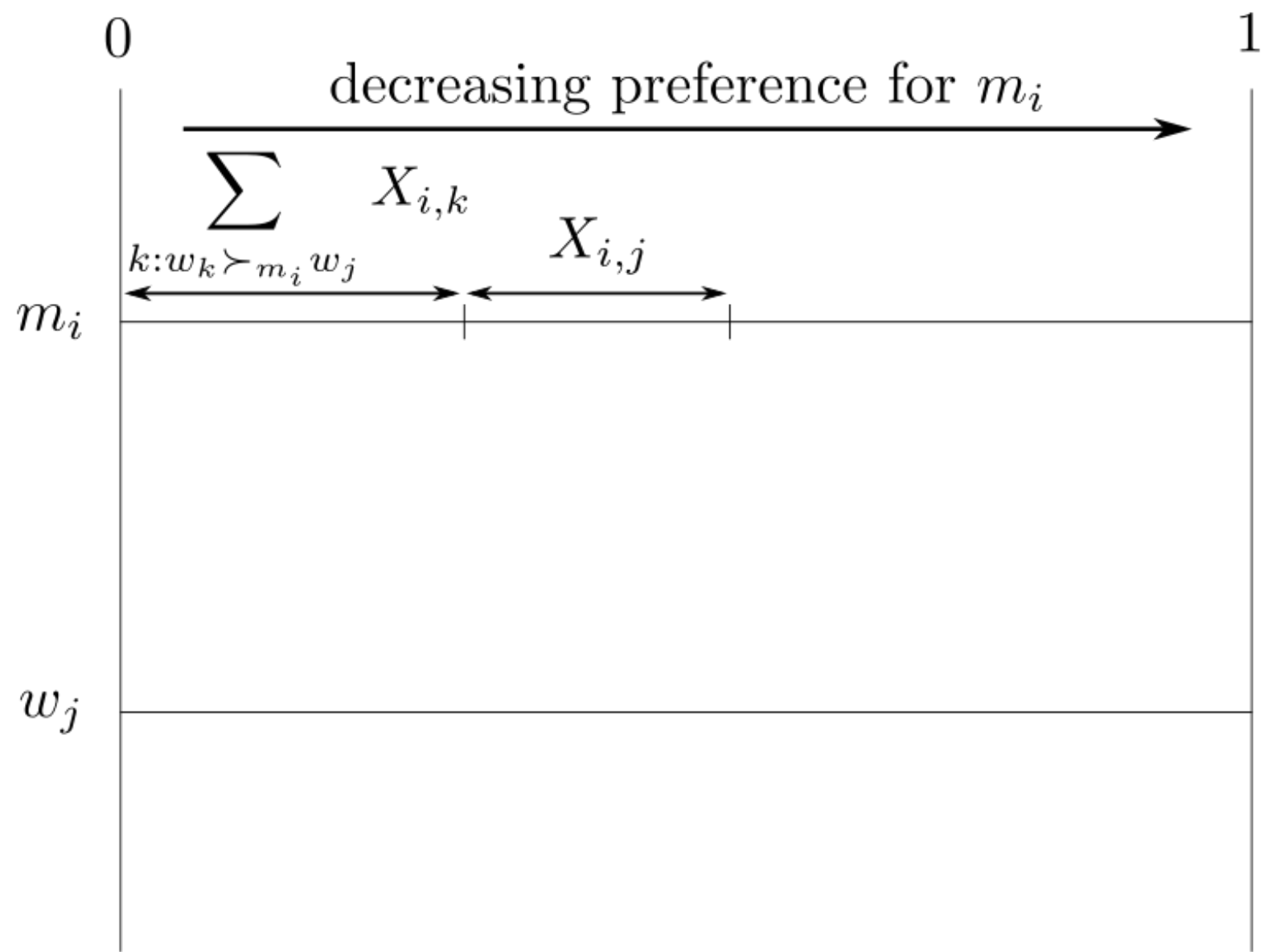


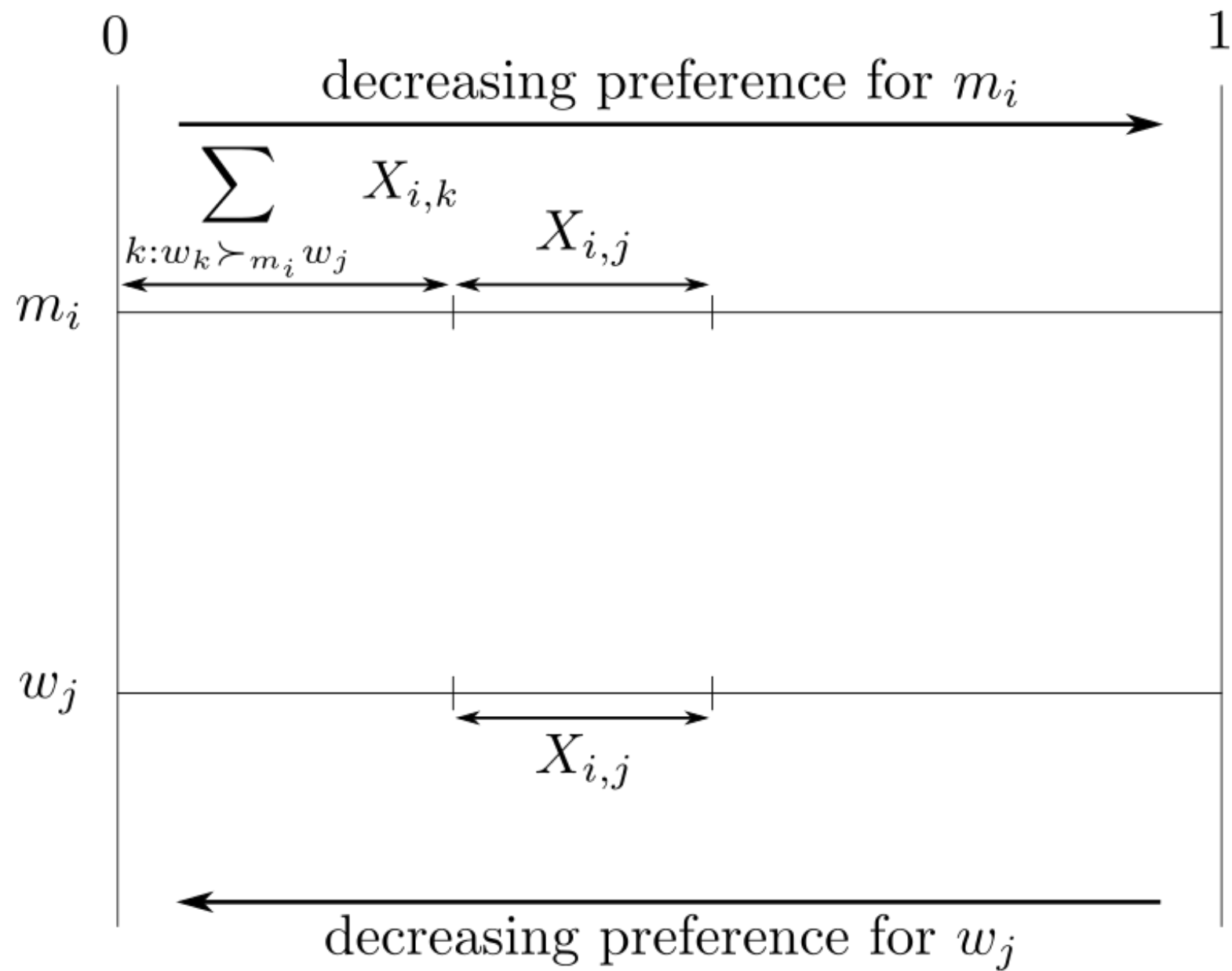


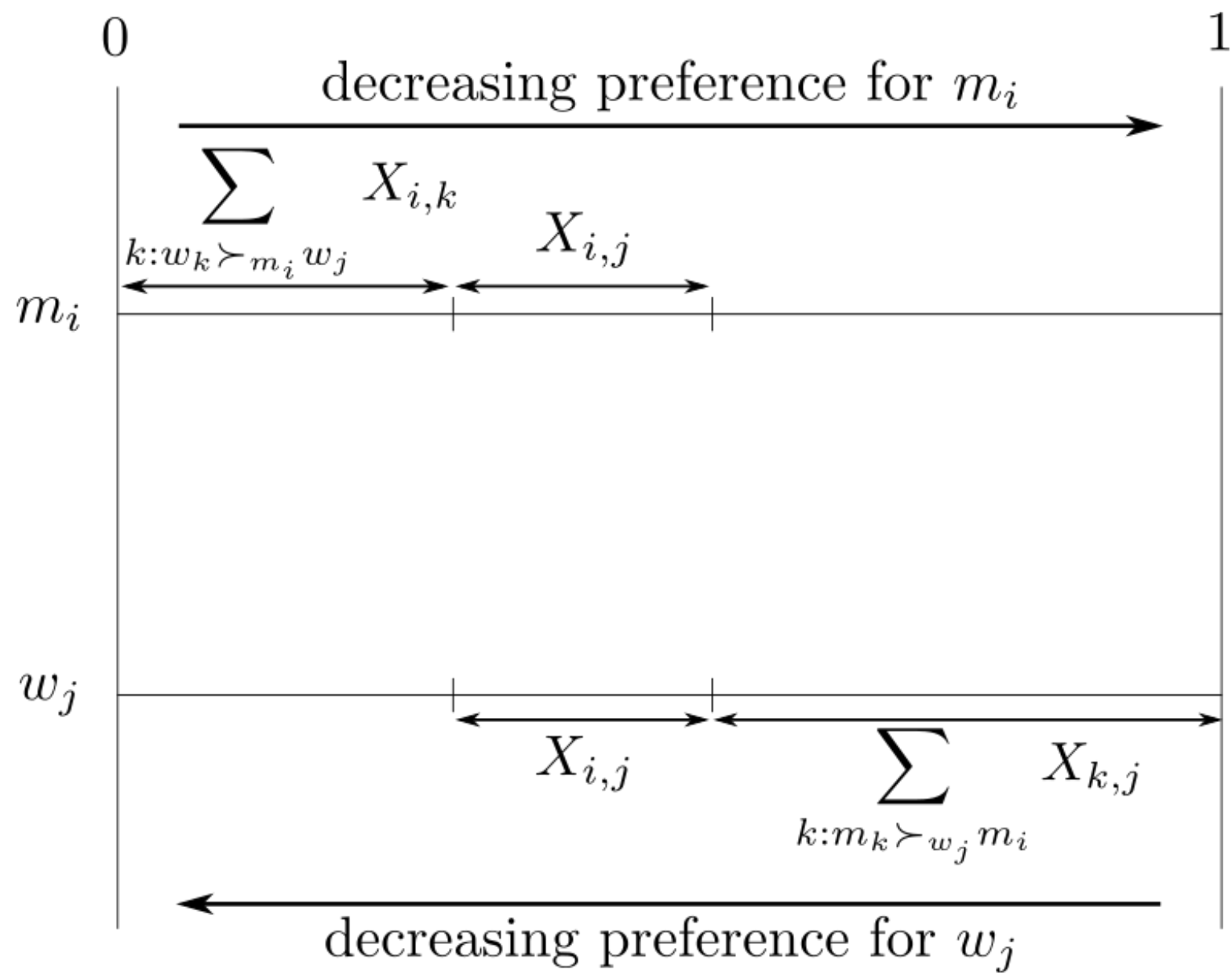


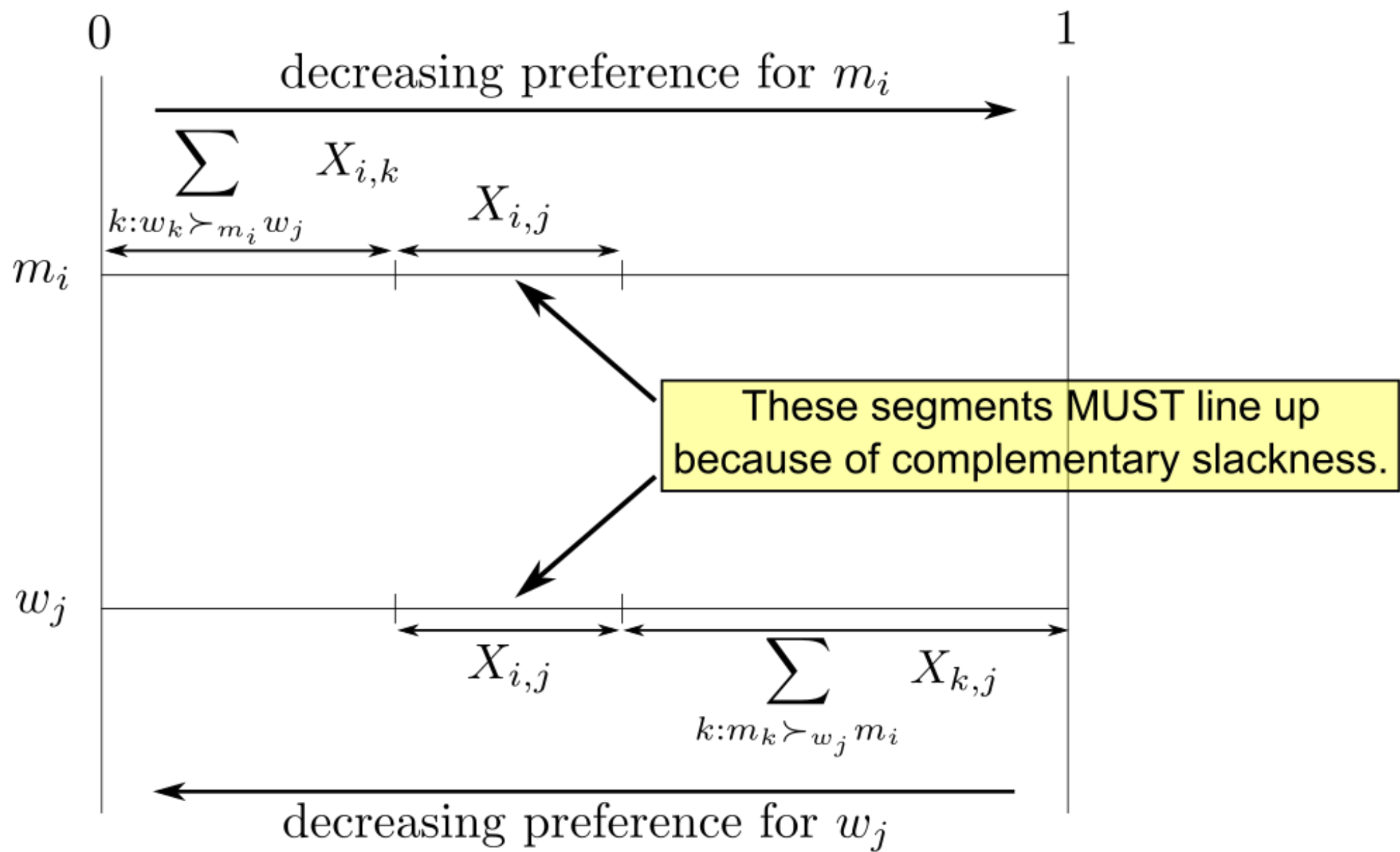


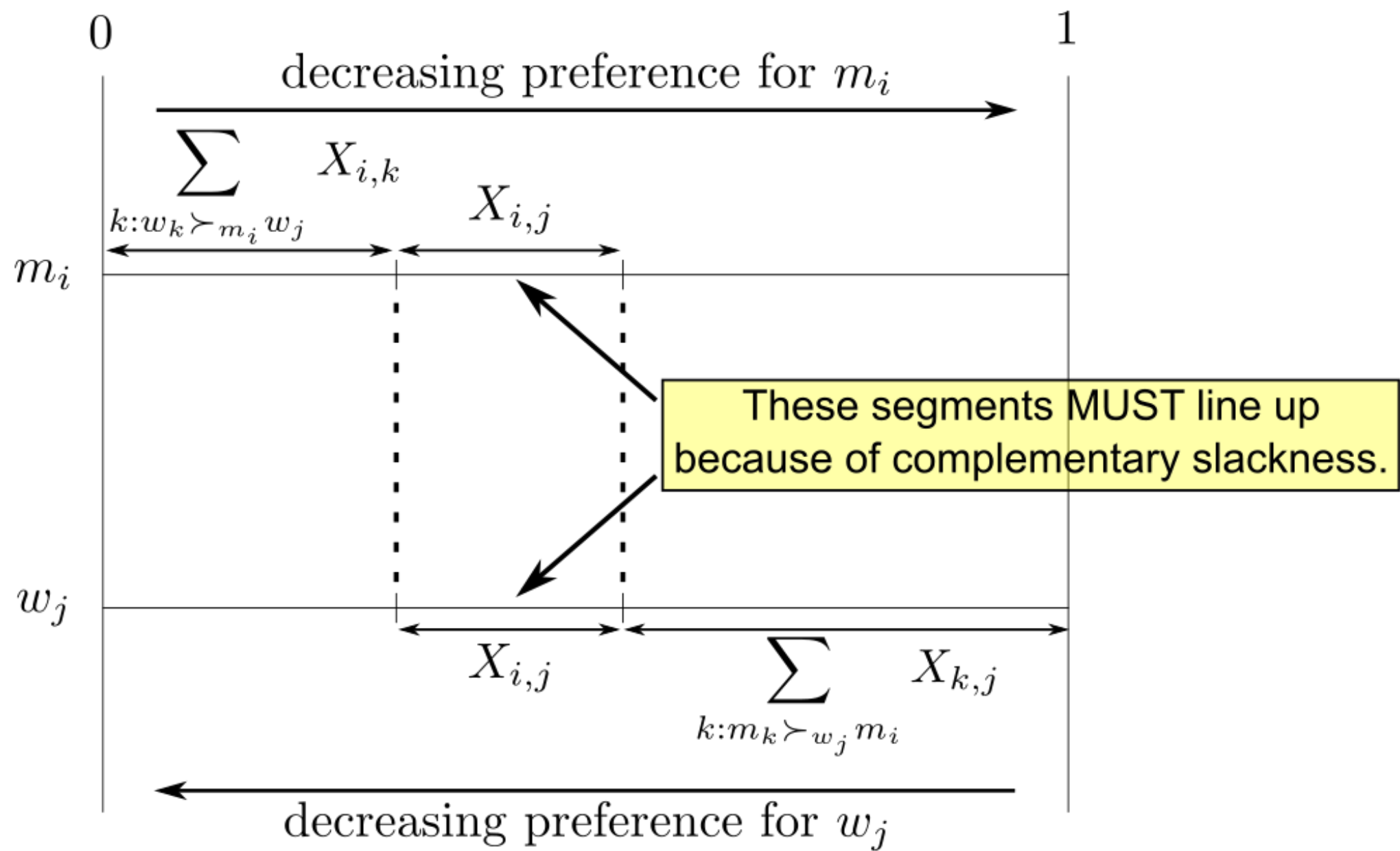


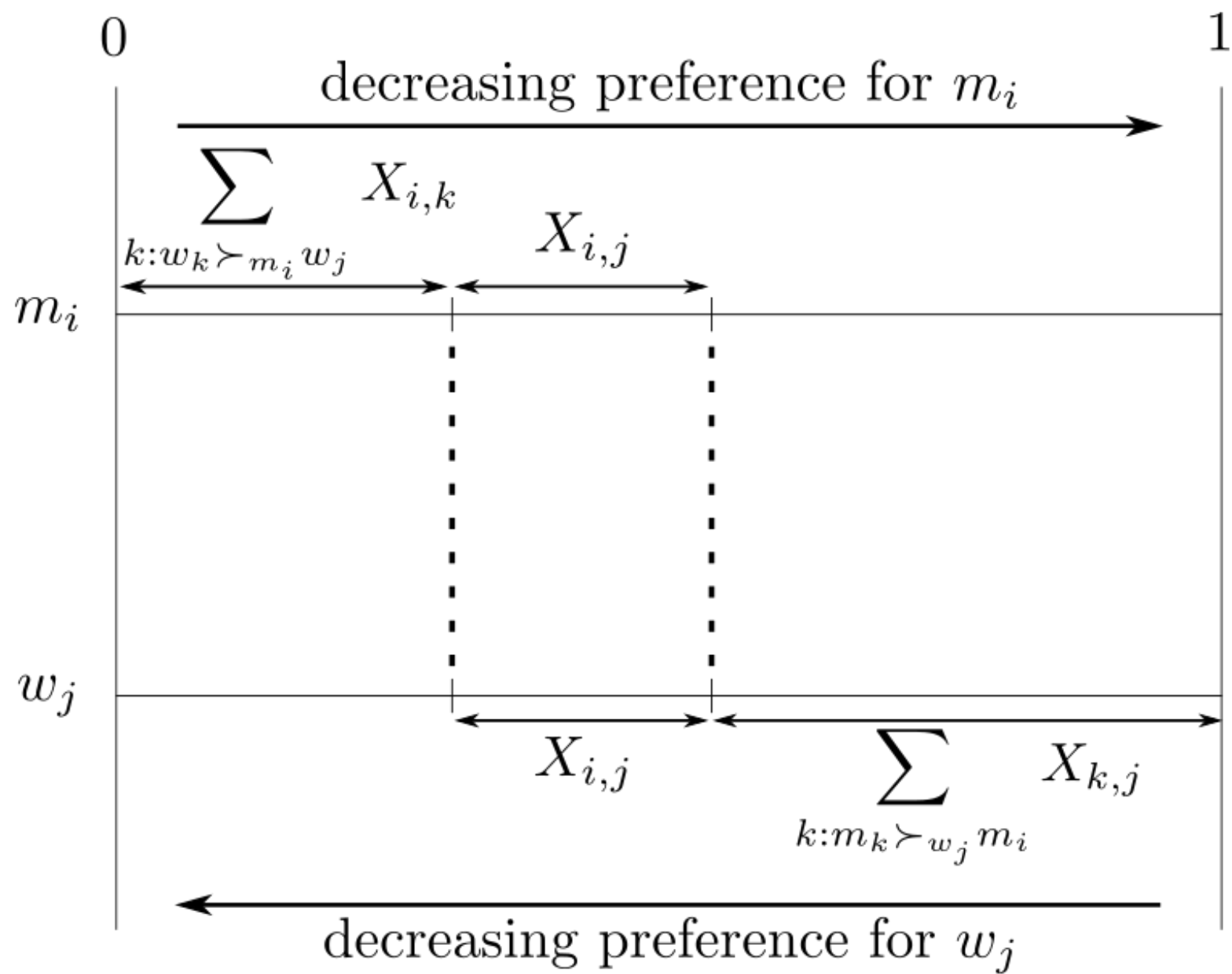


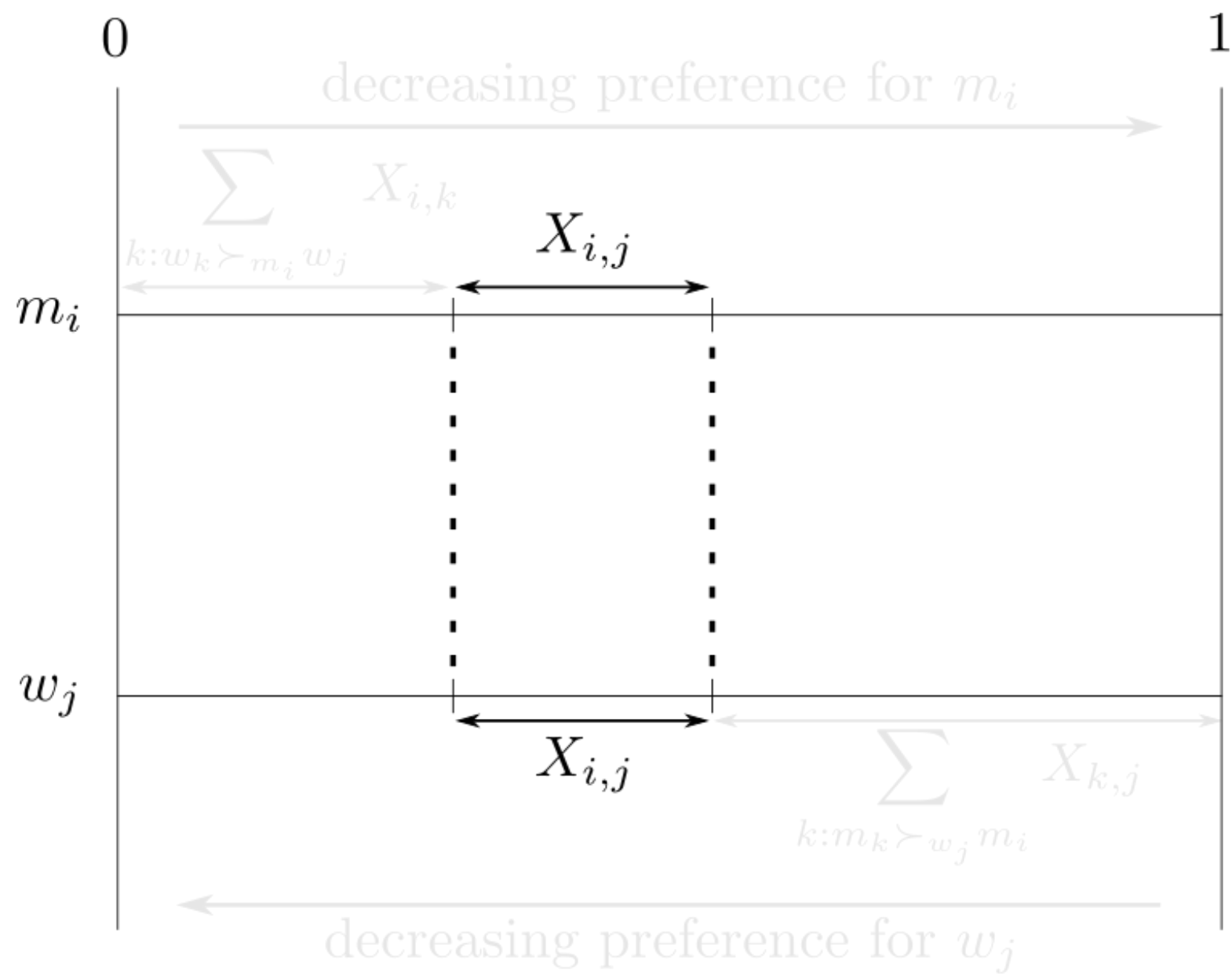


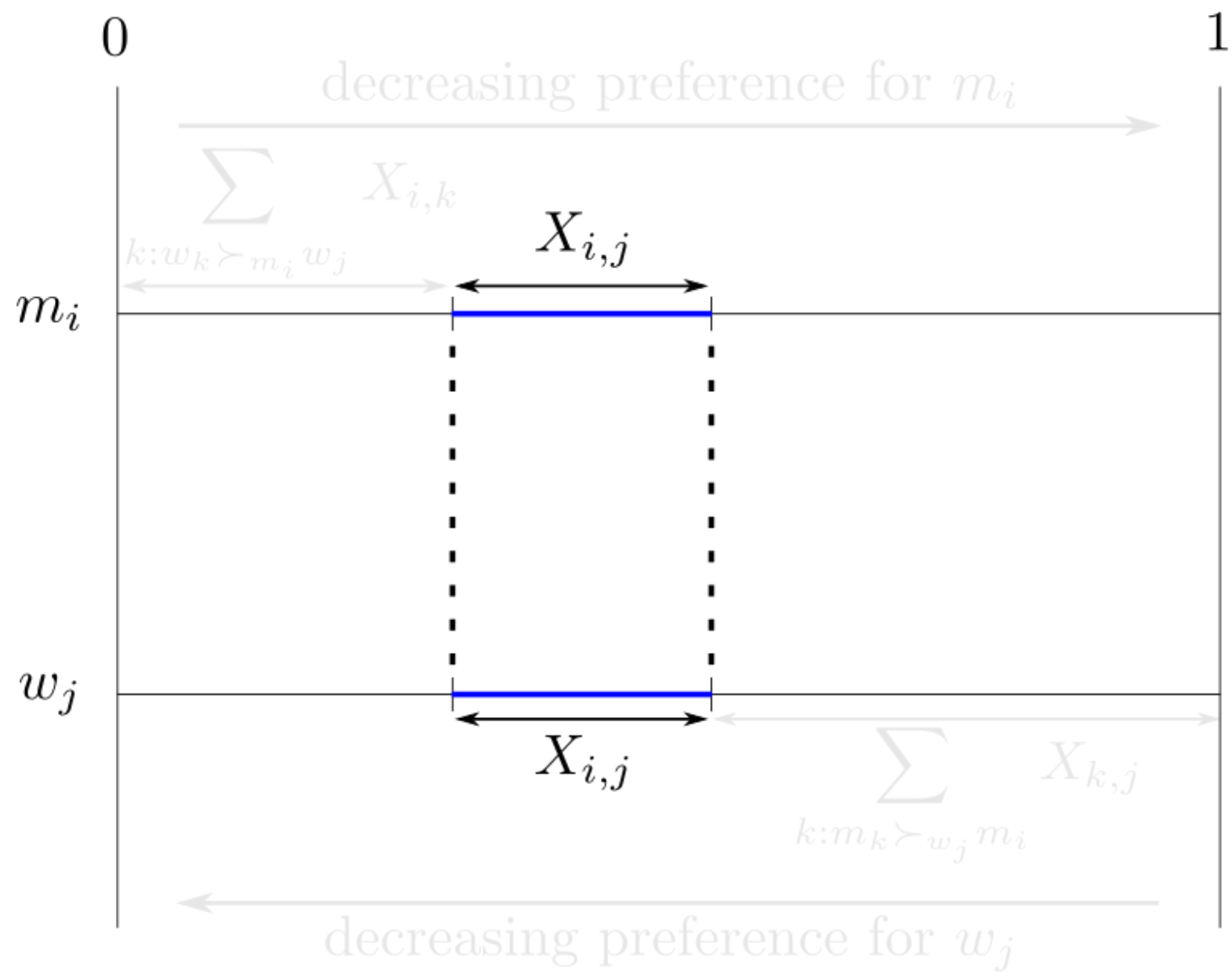


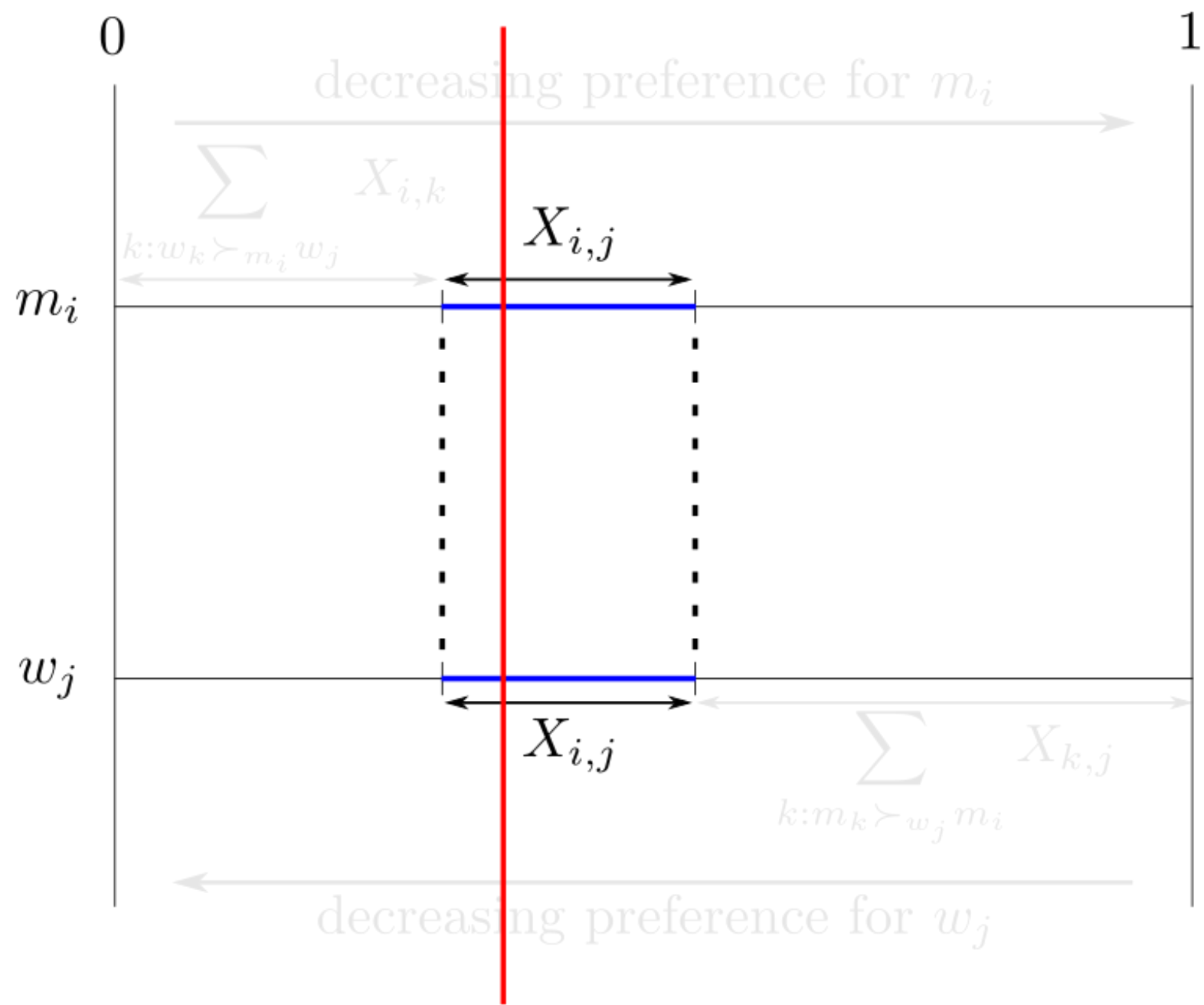


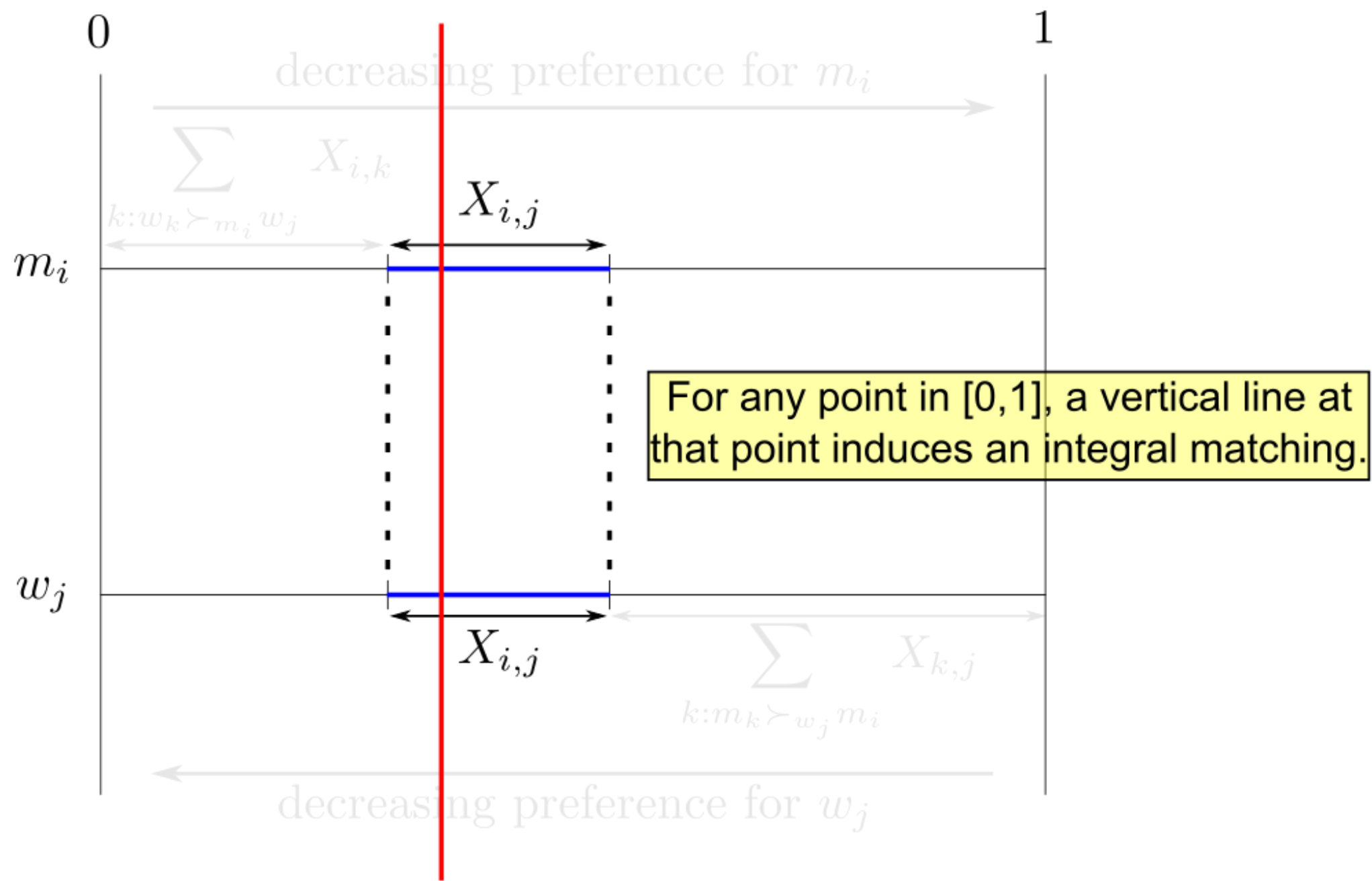


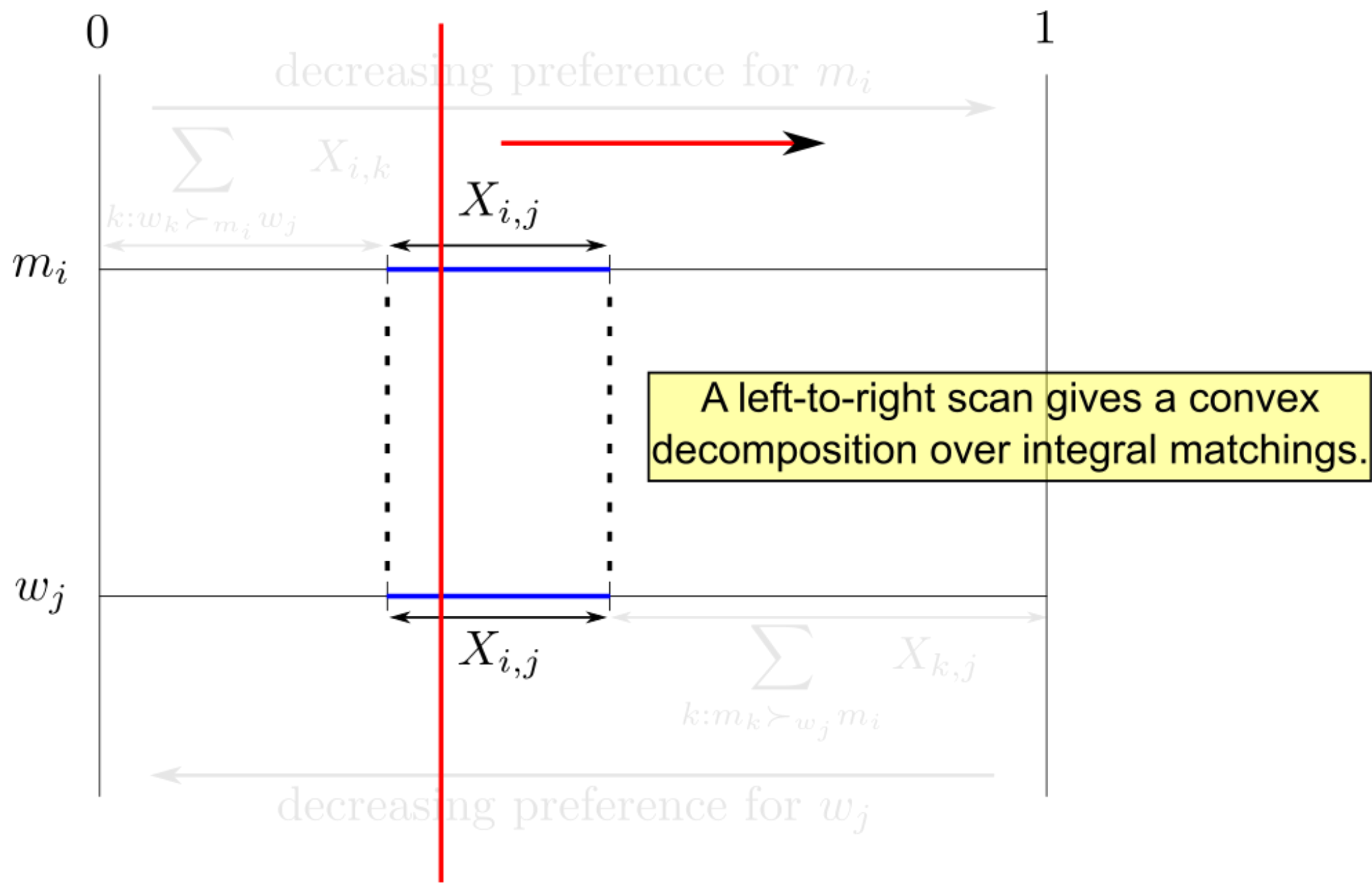


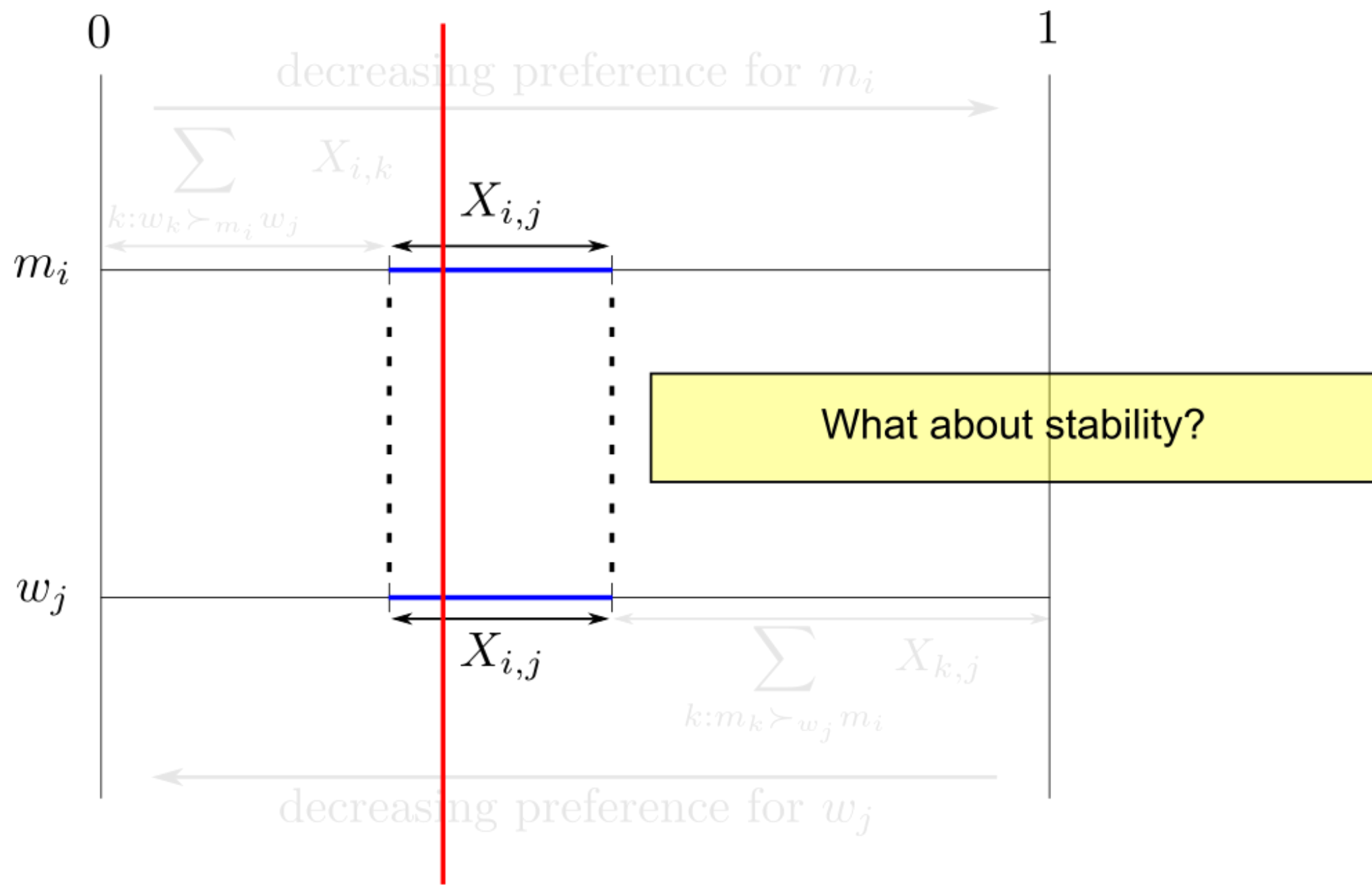


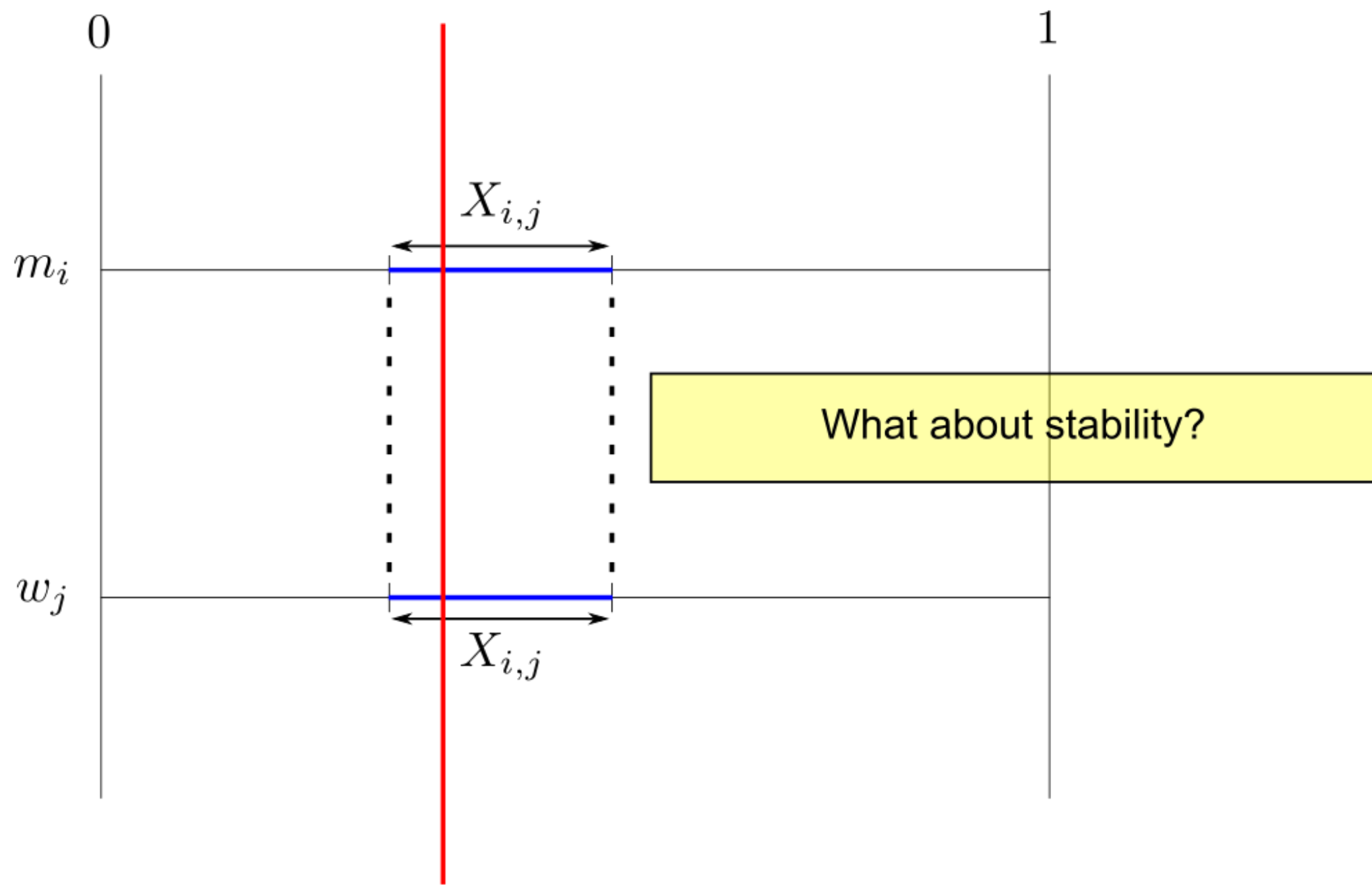


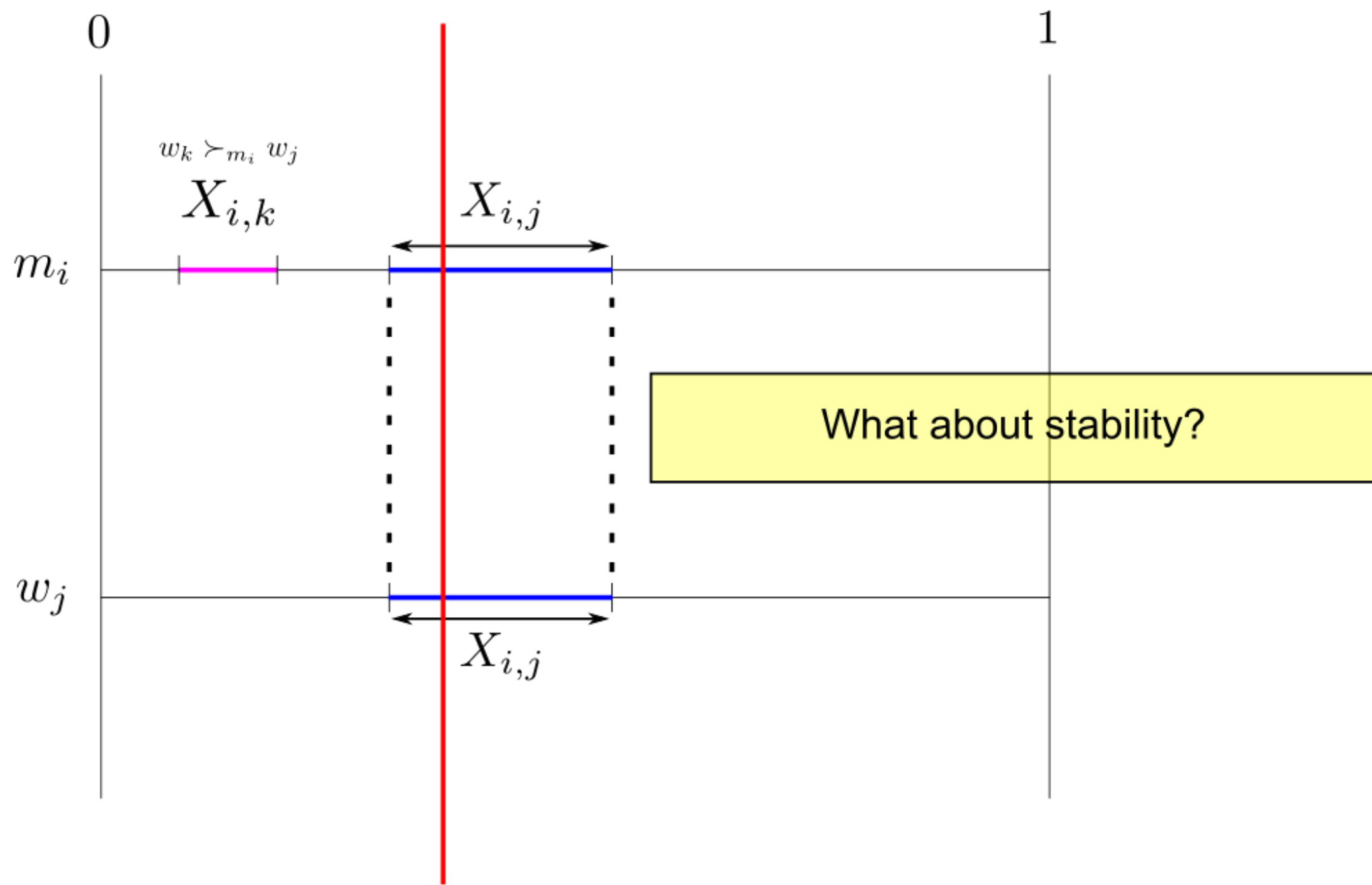


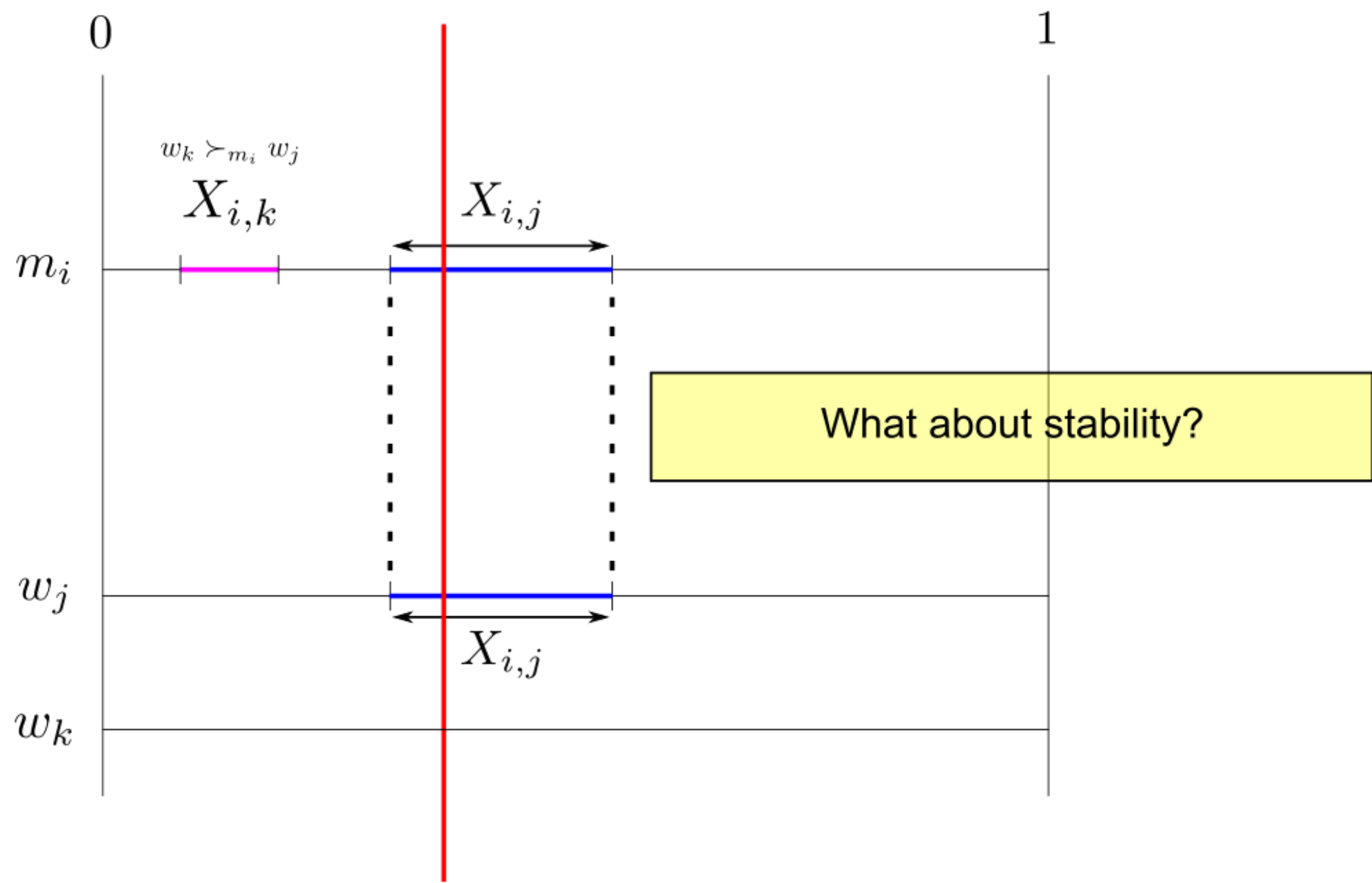


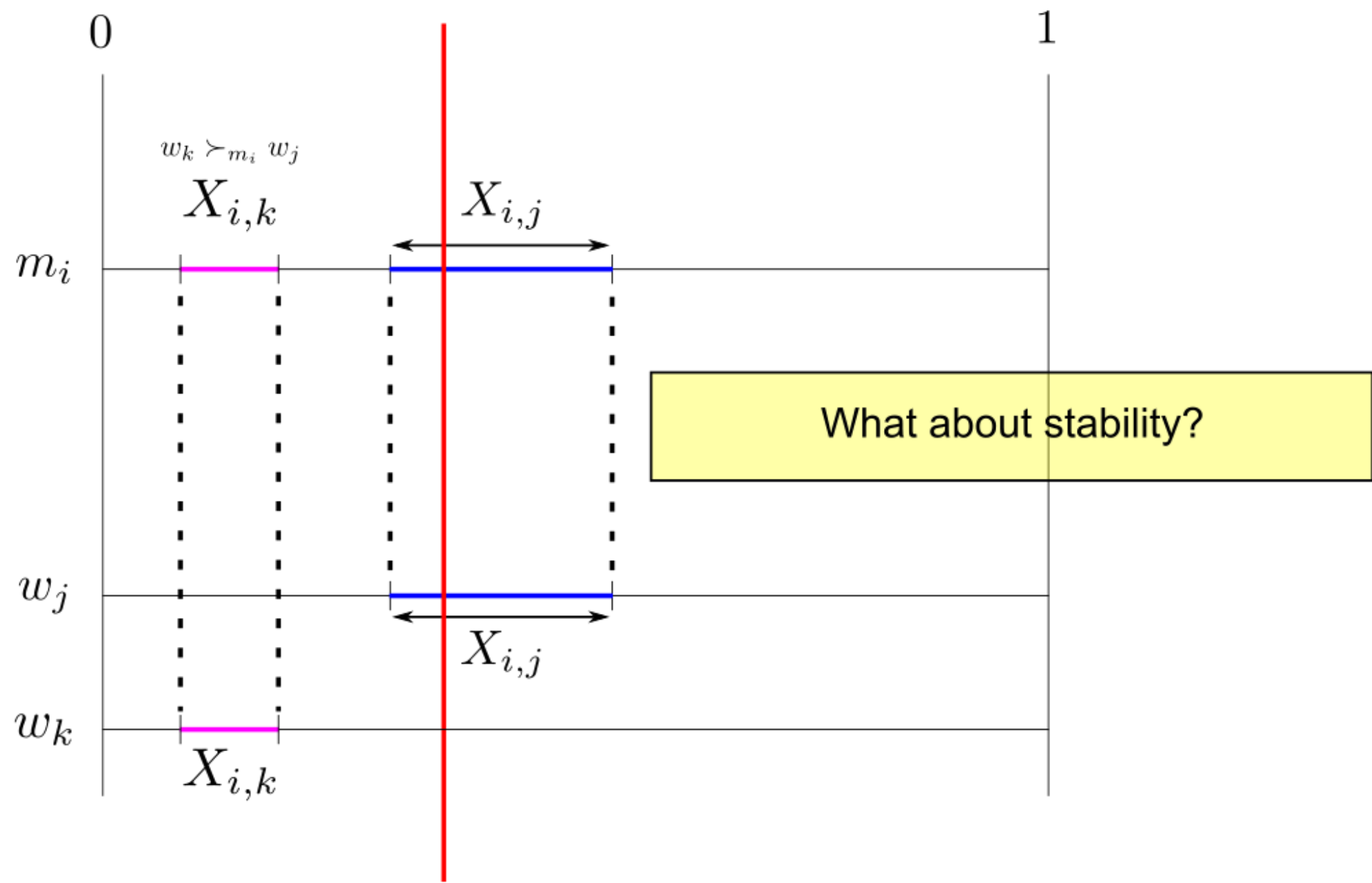


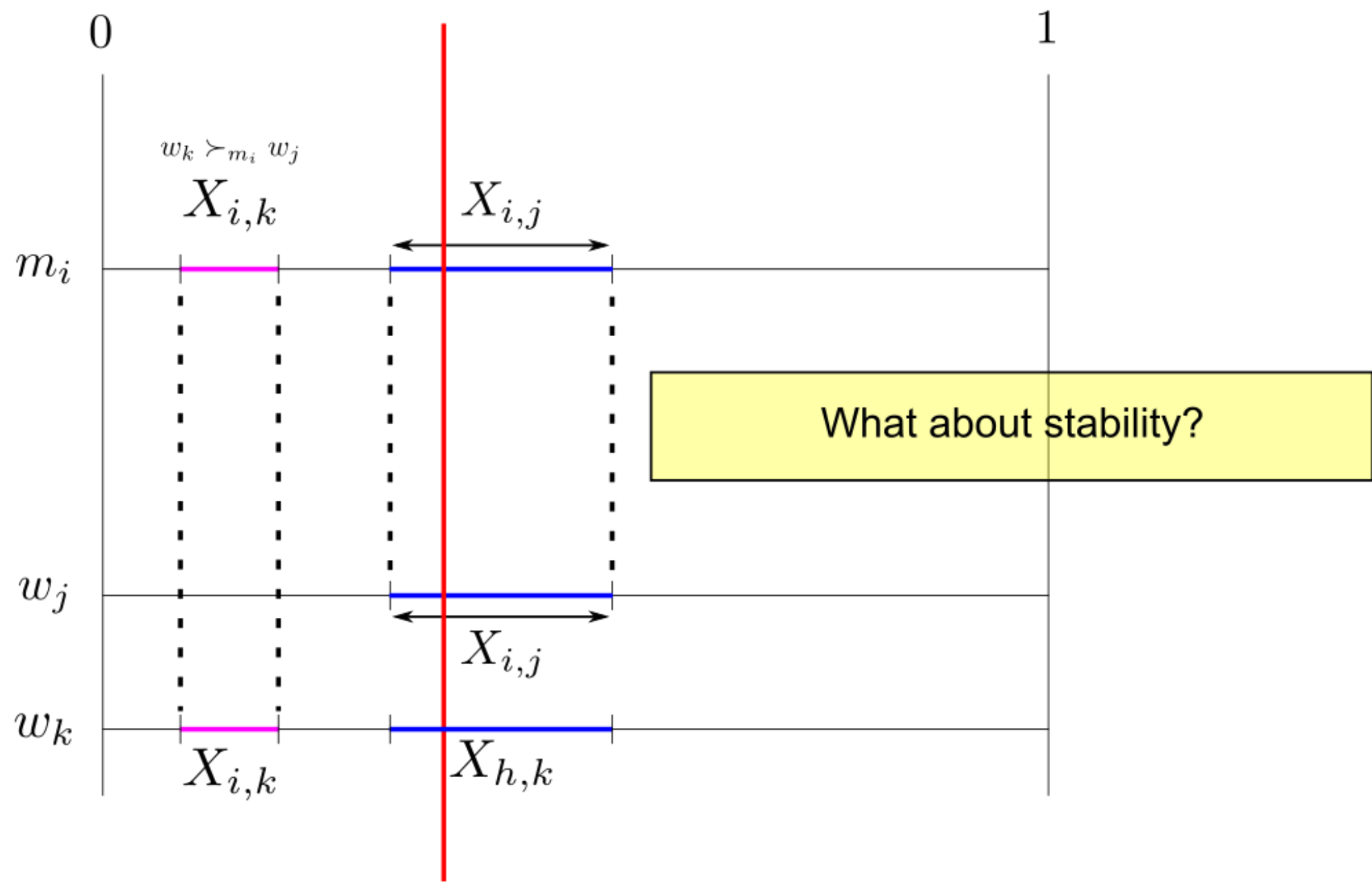


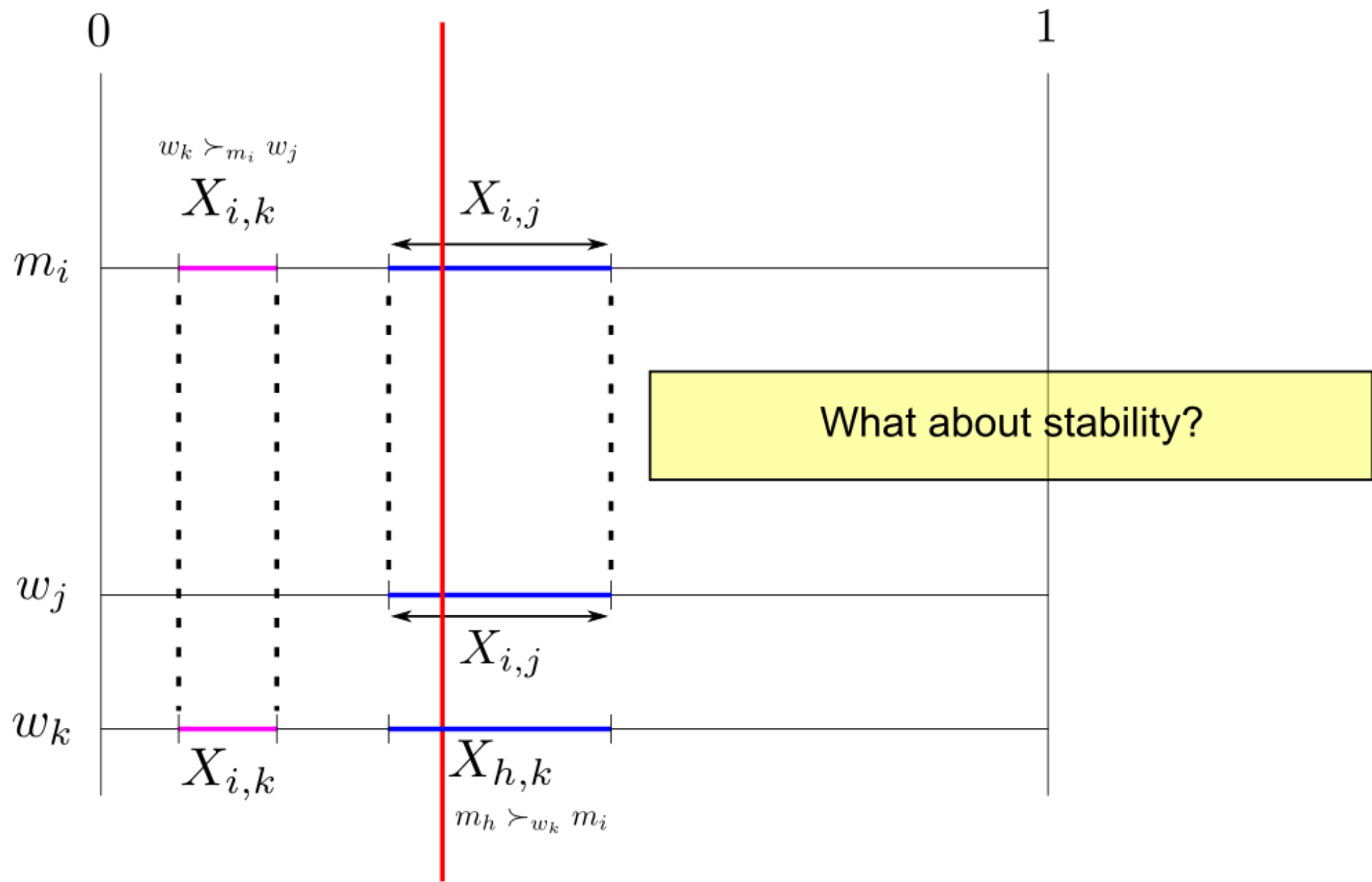


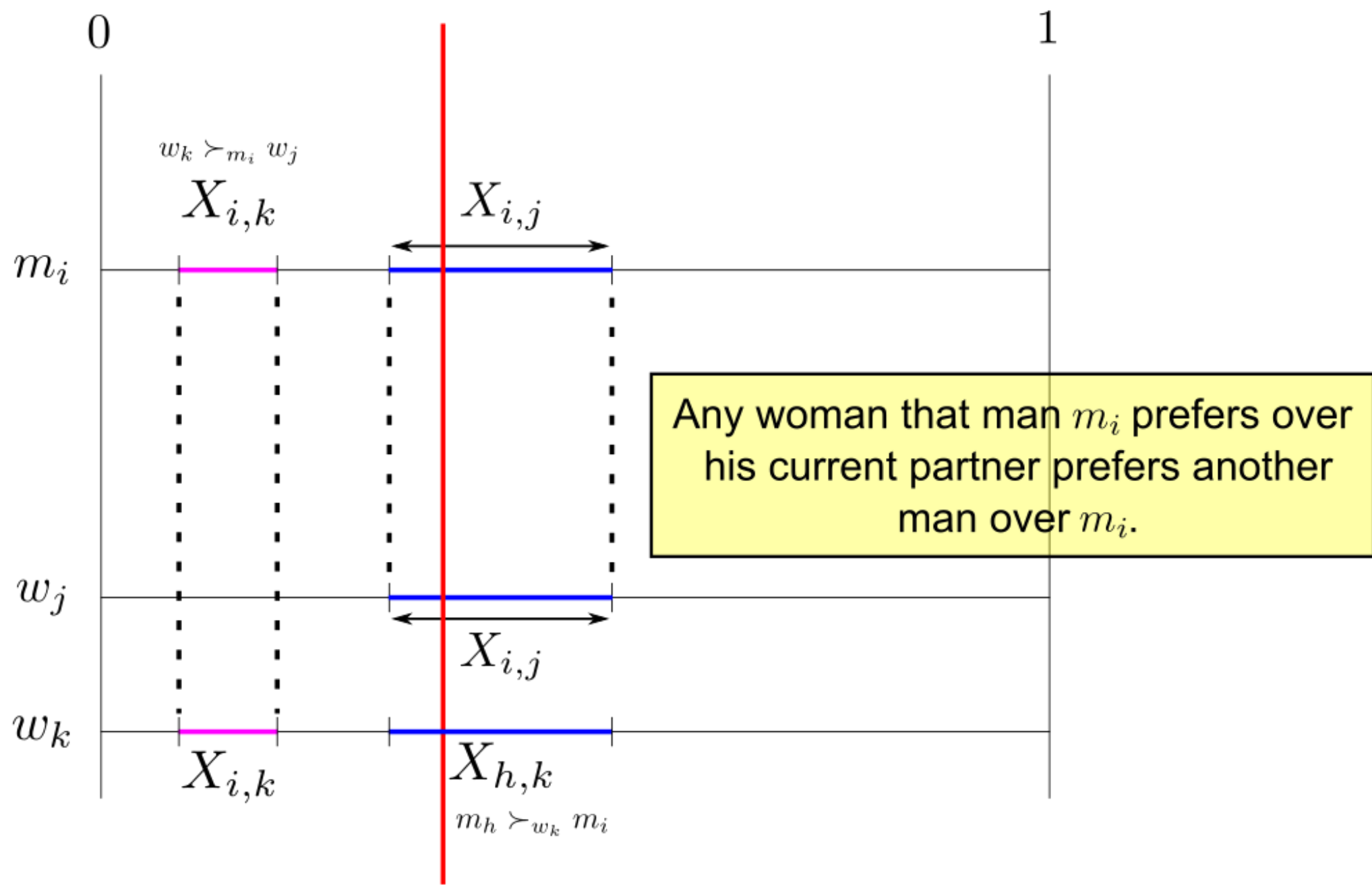


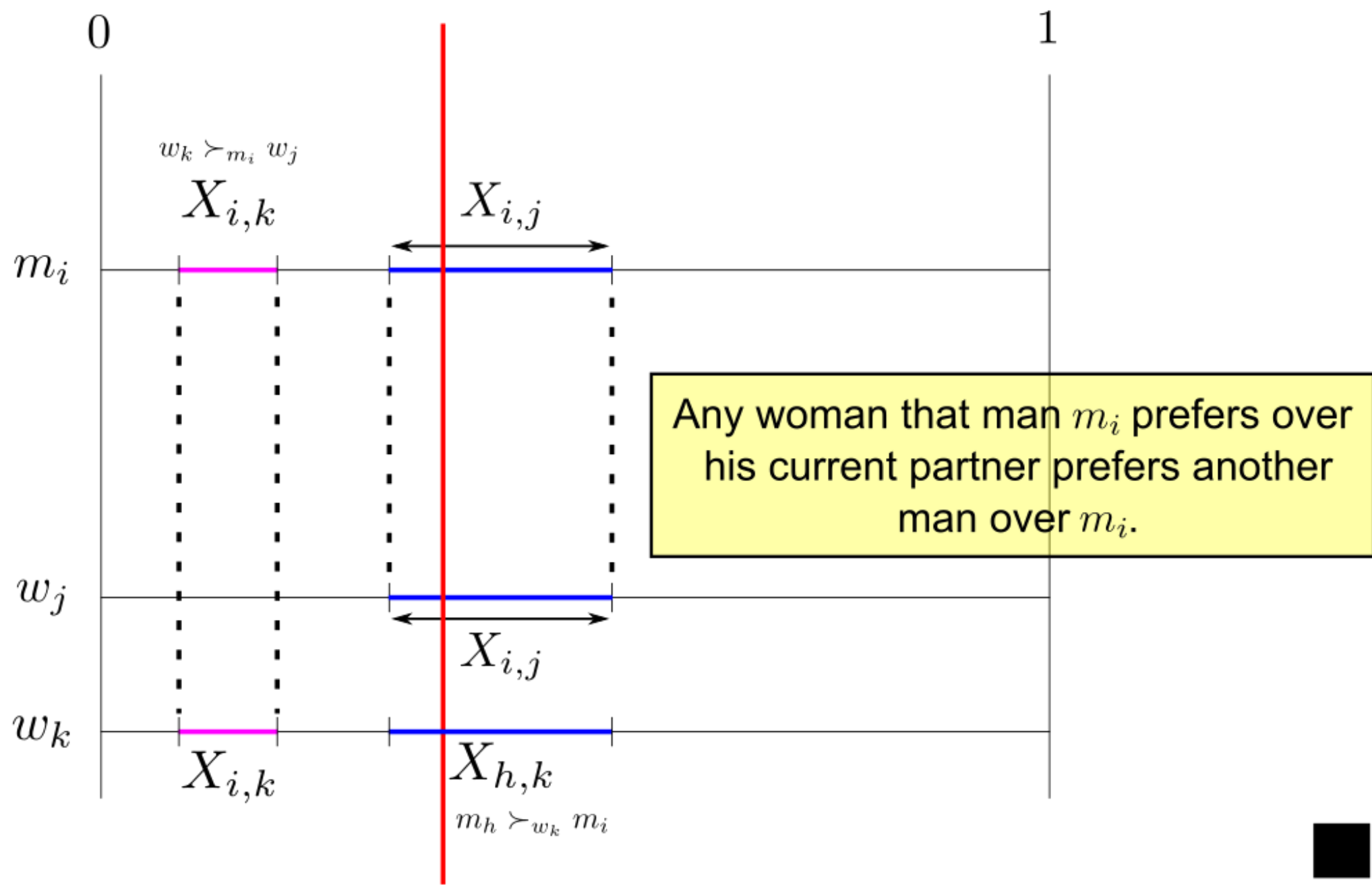












Let us use the decomposition technique to show the existence of a "fair" matching.

A "Fair" Stable Matching

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Suppose there are L stable matchings μ_1, \dots, μ_L for a given instance.

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For each man m_i , define his *median rank* as:

$$\text{med}(m_i) = \text{median}(\text{rank of } \mu_1(m_i) \text{ in } \succ_{m_i}, \dots, \text{rank of } \mu_L(m_i) \text{ in } \succ_{m_i}).$$

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Median mapping: Each agent points to its median rank agent.

1,1,1,1 | 4,4,3,3

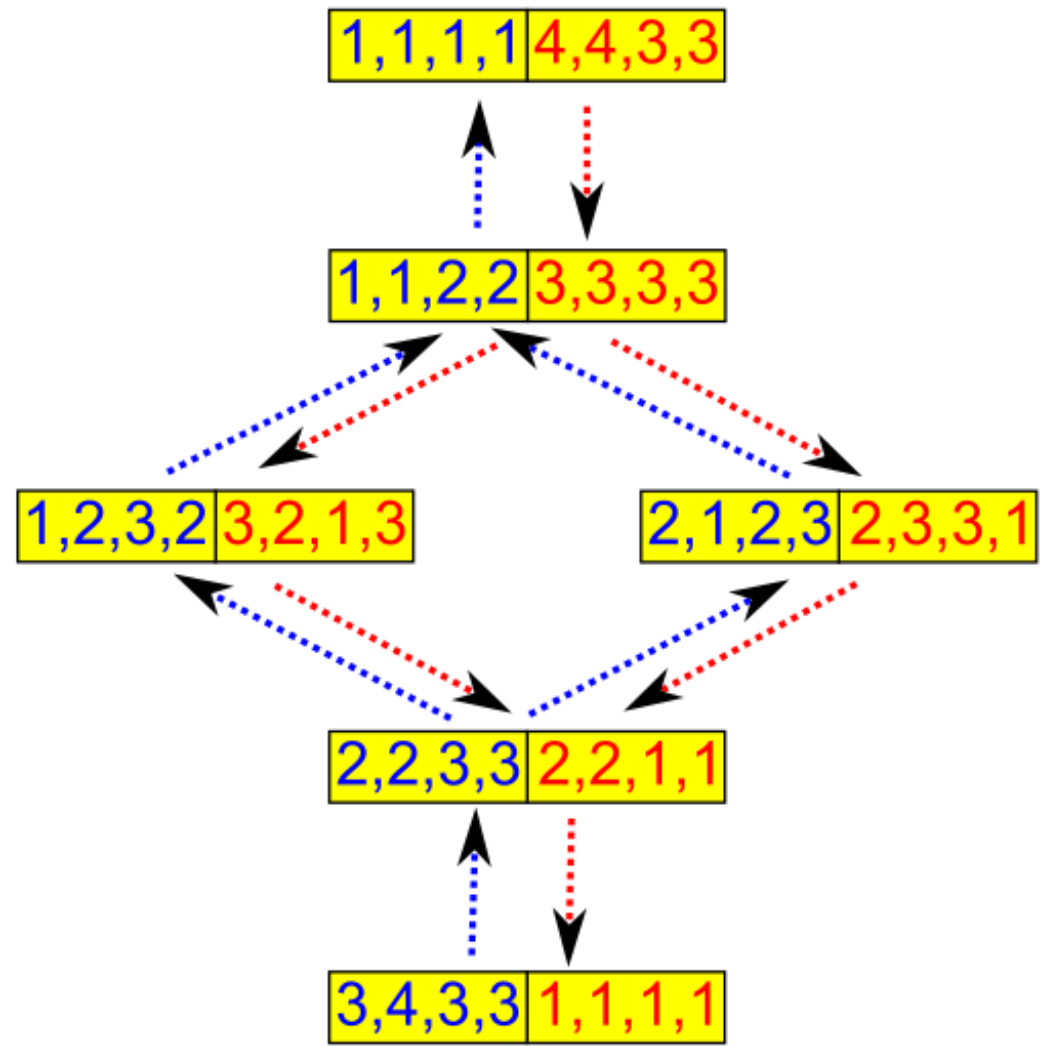
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

3,4,3,3 | 1,1,1,1



1,1,1,1 | 4,4,3,3

m_1 1 1 1 2 2 3

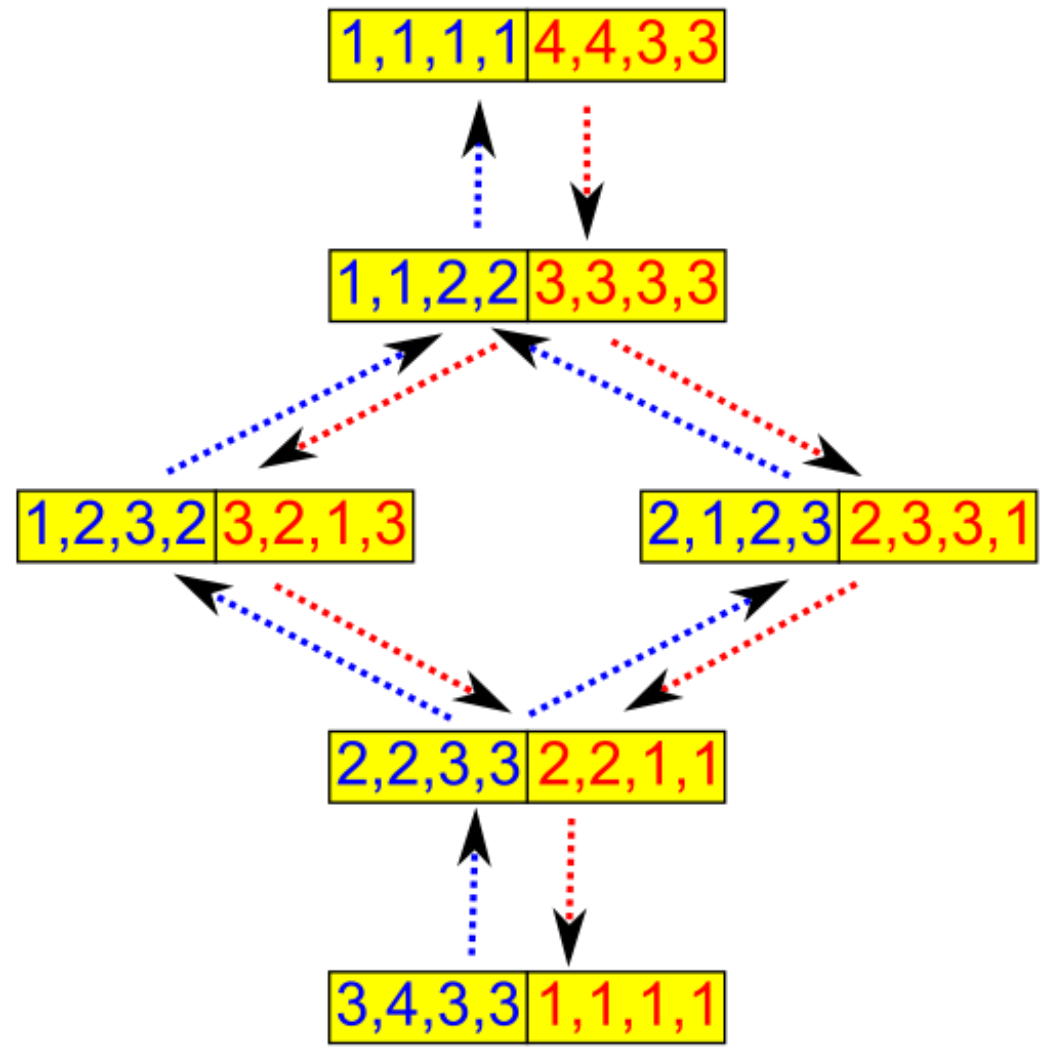
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

3,4,3,3 | 1,1,1,1



1,1,1,1 | 4,4,3,3

1,1,2,2 | 3,3,3,3

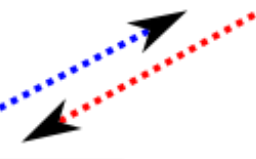
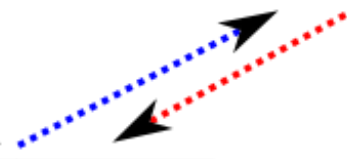
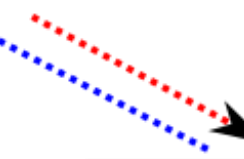
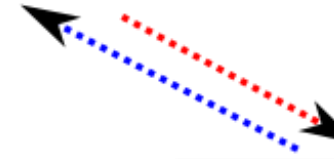
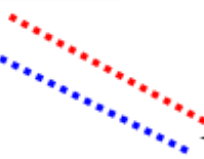
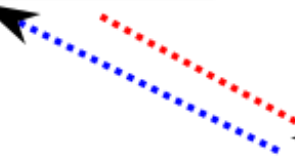
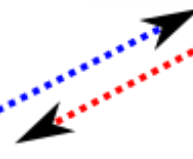
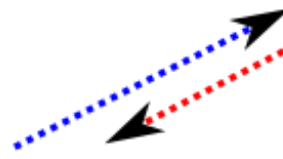
1,2,3,2 | 3,2,1,3

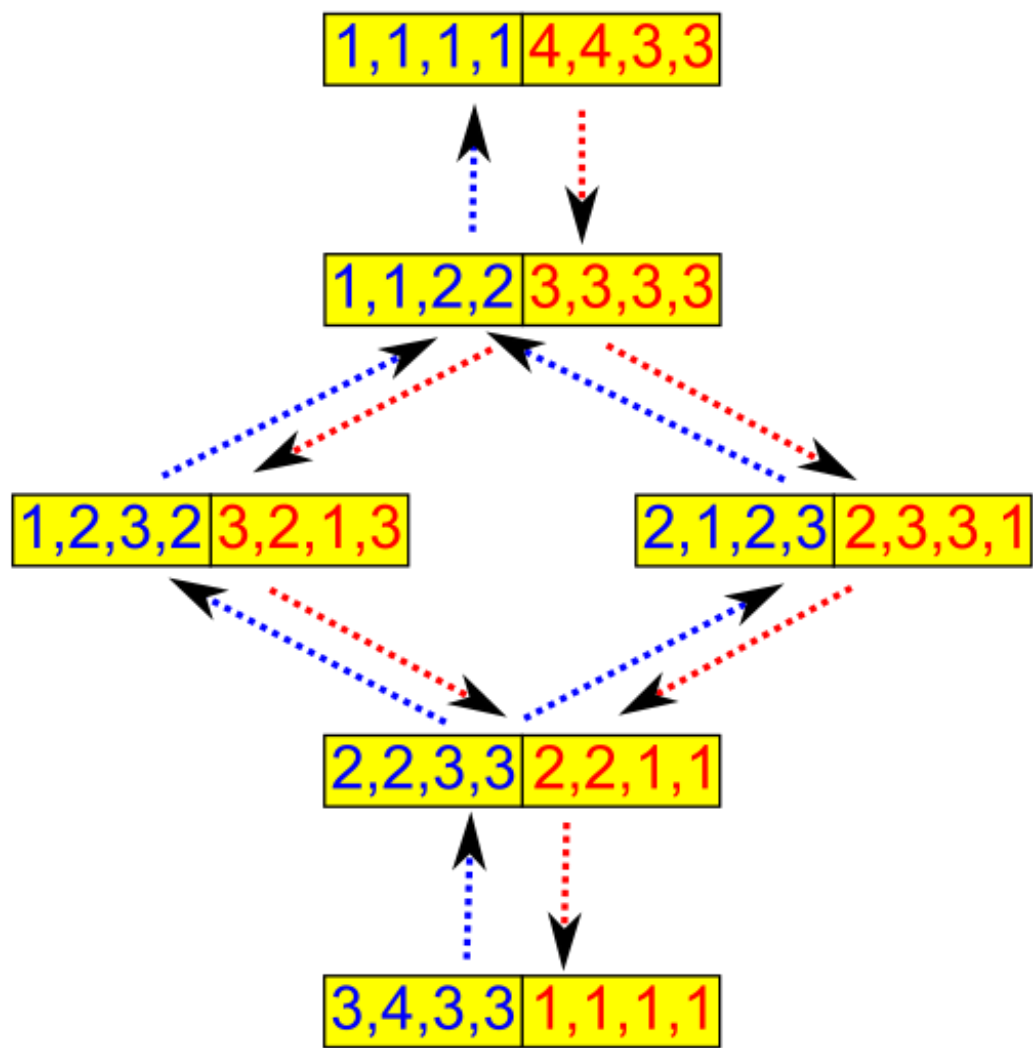
2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

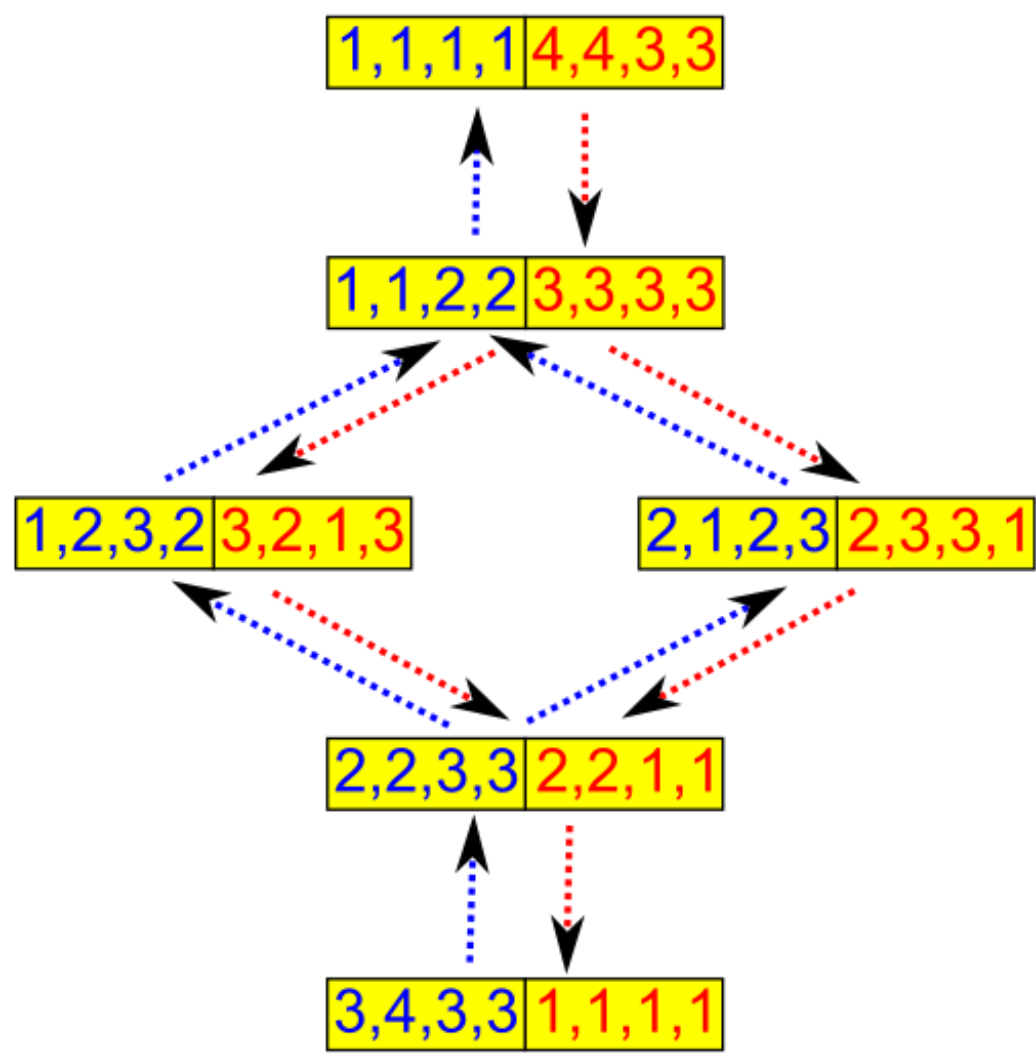
3,4,3,3 | 1,1,1,1

| | | | | | | |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |

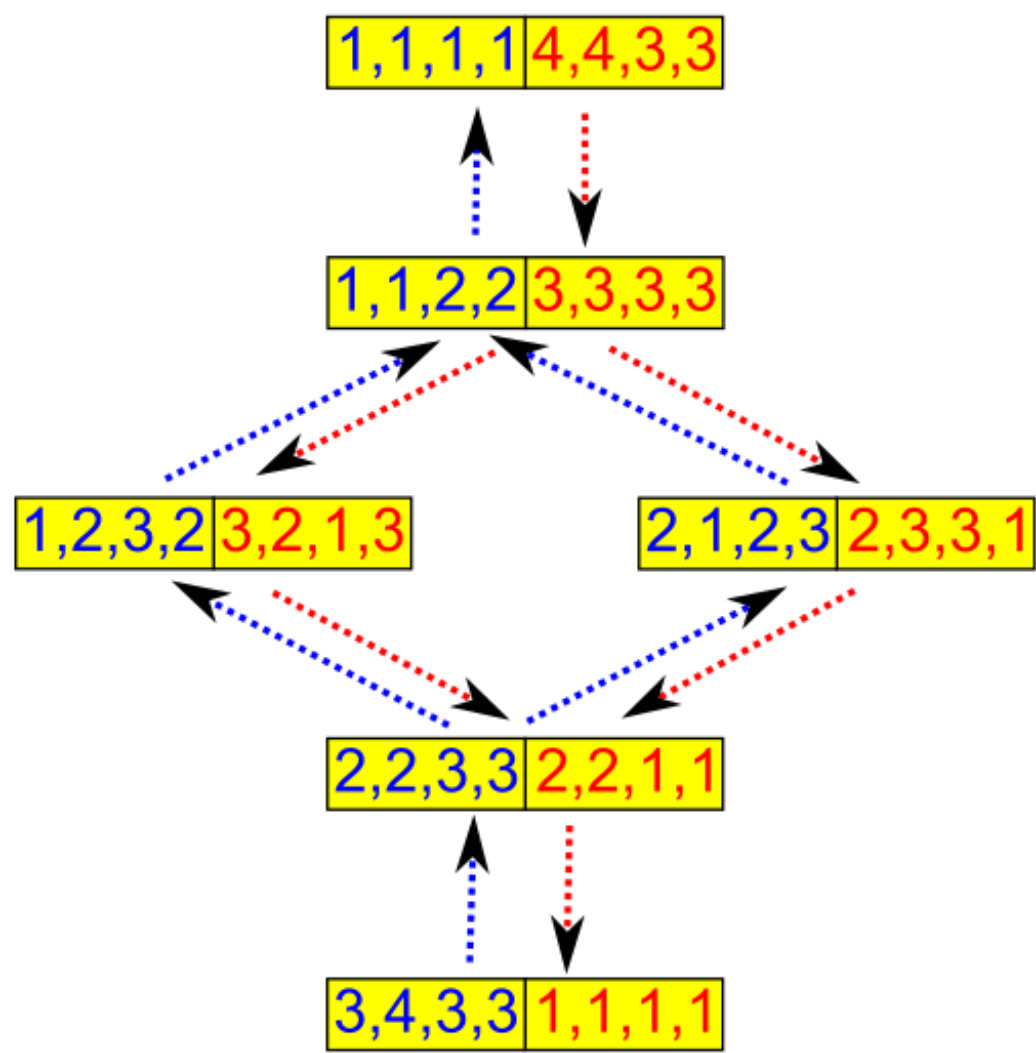




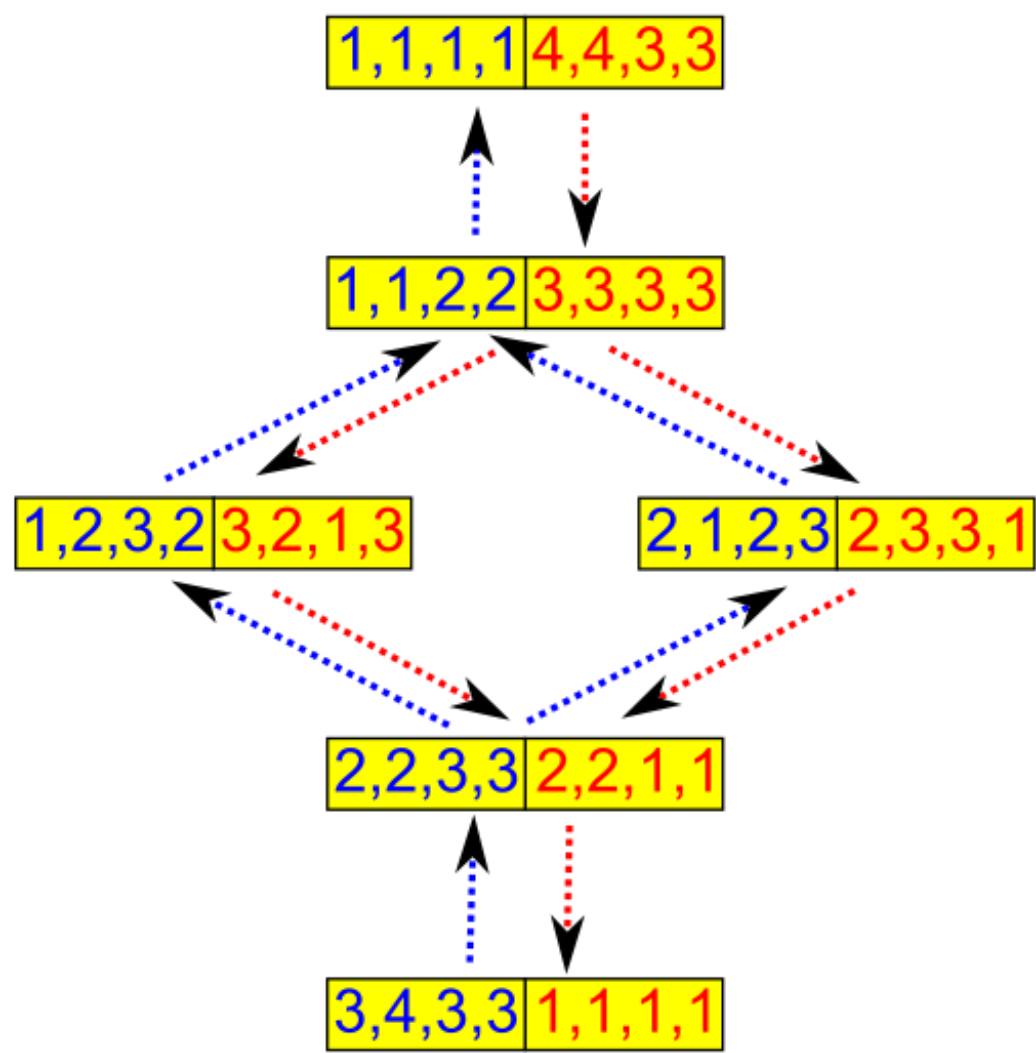
| | | | | | | |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |



| | | | | | | |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |
| w_1 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_2 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_3 | 3 | 3 | 3 | 1 | 1 | 1 |
| w_4 | 3 | 3 | 3 | 1 | 1 | 1 |



| | | | | | | |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |
| w_1 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_2 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_3 | 3 | 3 | 3 | 1 | 1 | 1 |
| w_4 | 3 | 3 | 3 | 1 | 1 | 1 |



| | | | | | | |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |
| w_1 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_2 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_3 | 3 | 3 | 3 | 1 | 1 | 1 |
| w_4 | 3 | 3 | 3 | 1 | 1 | 1 |

A "Fair" Stable Matching

Suppose there are L stable matchings μ_1, \dots, μ_L for a given instance.

For each man m_i , define his *median rank* as:

$$\text{med}(m_i) = \text{median}(\text{rank of } \mu_1(m_i) \text{ in } \succ_{m_i}, \dots, \text{rank of } \mu_L(m_i) \text{ in } \succ_{m_i}).$$

For each woman w_j , define her *median rank* as:

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Median mapping: Each agent points to its median rank agent.

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[Teo and Sethuraman, *MOR* 1998]

The median mapping induces a stable matching.

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[Teo and Sethuraman, *MOR* 1998]

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Proof by example.

1,1,1,1 | 4,4,3,3

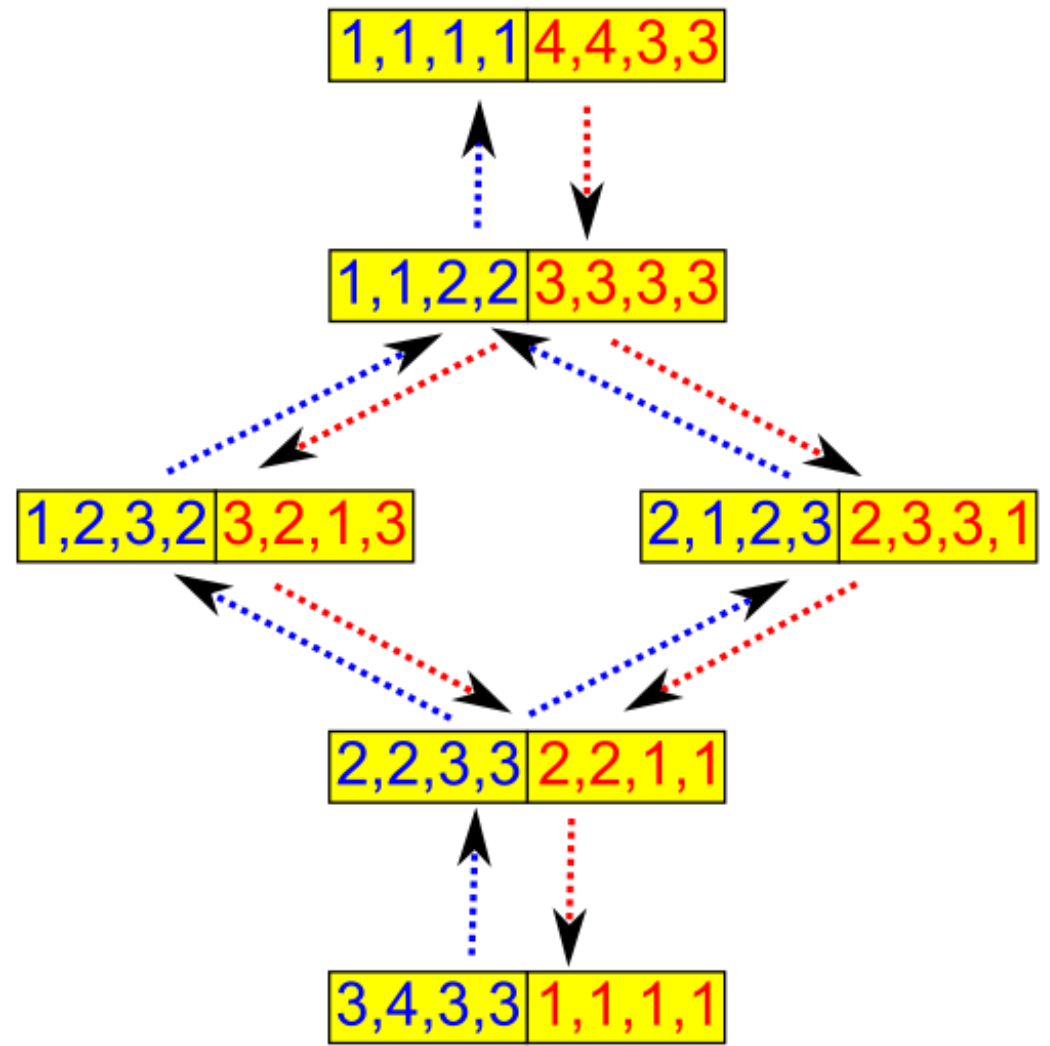
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

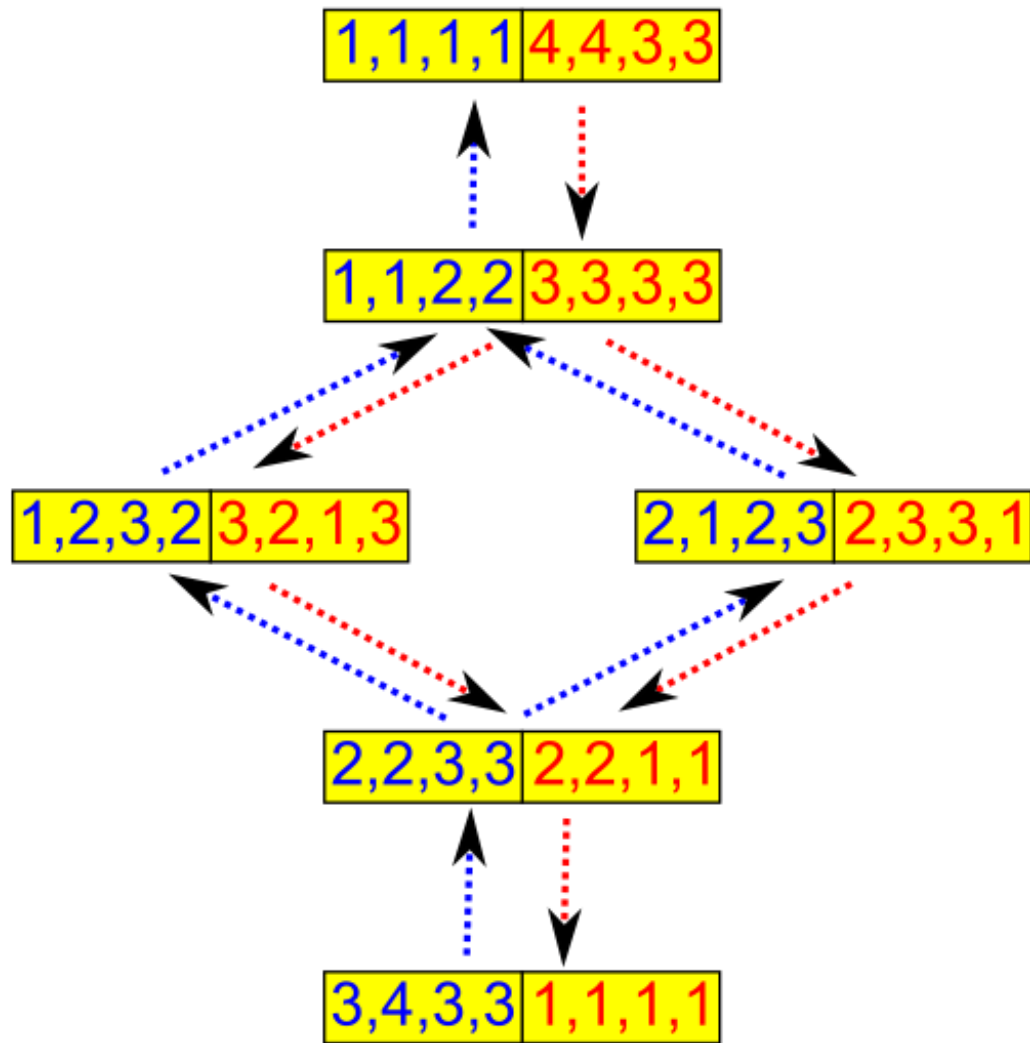
2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

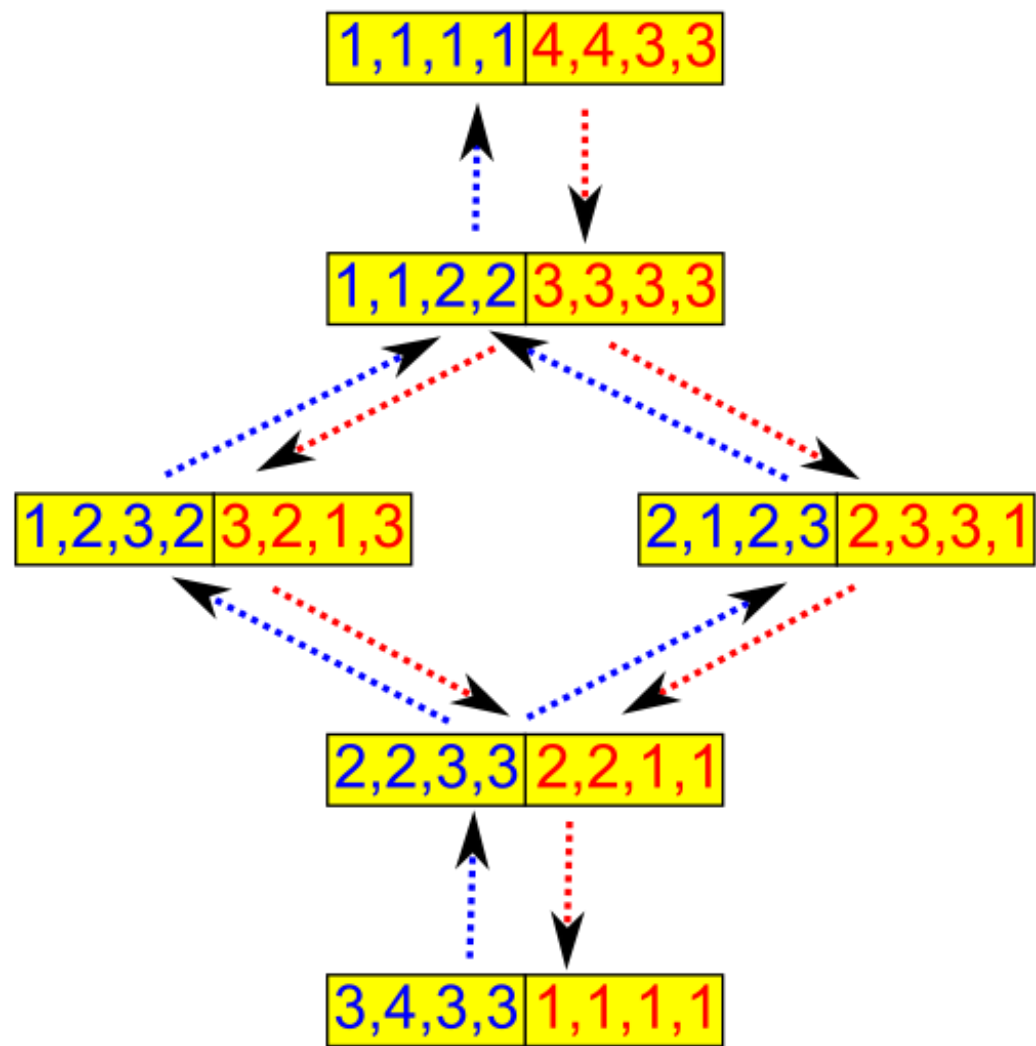
3,4,3,3 | 1,1,1,1



Consider a uniform combination of all integral stable matchings.



Consider a uniform combination of all integral stable matchings.



m_1

1

1

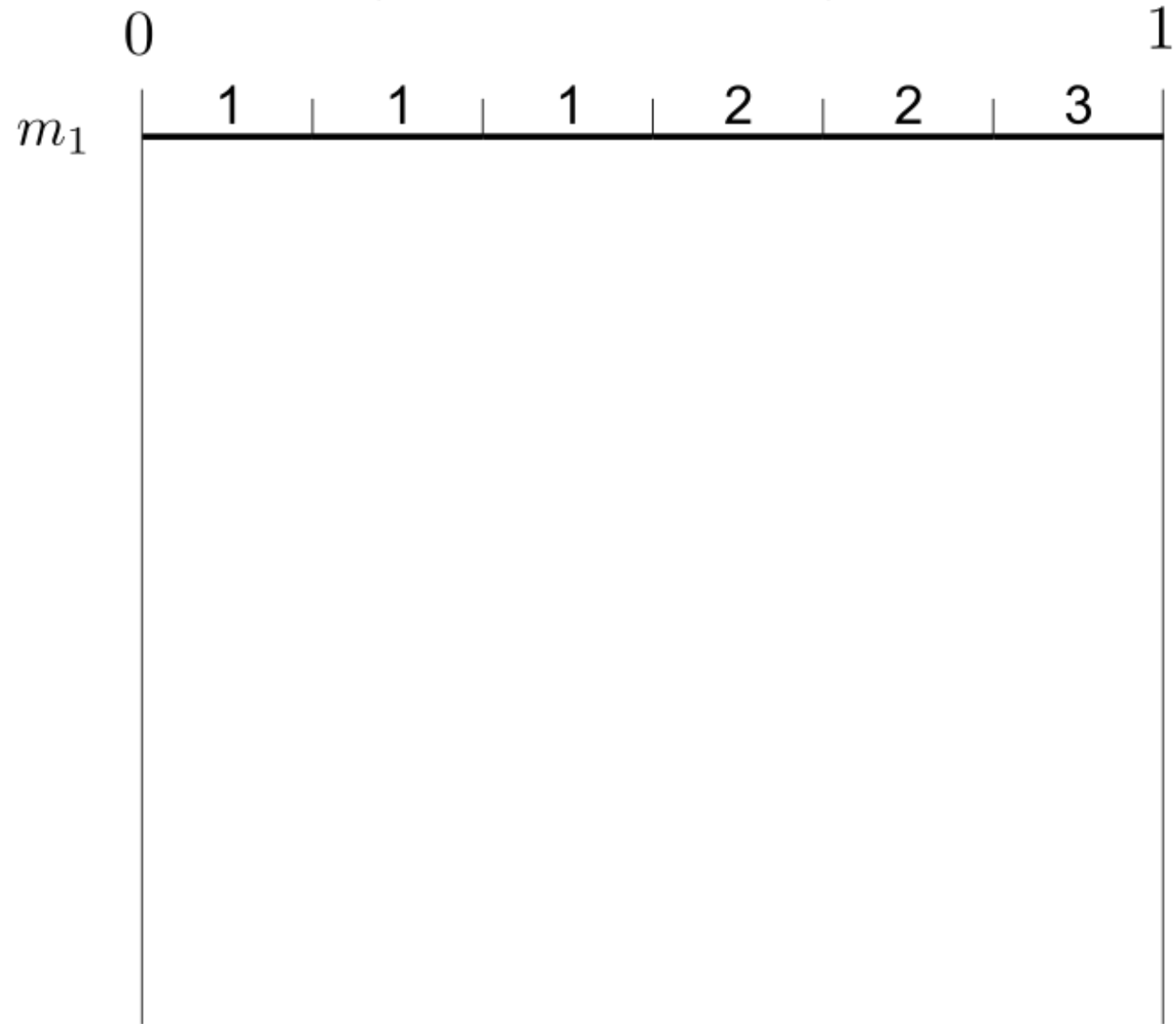
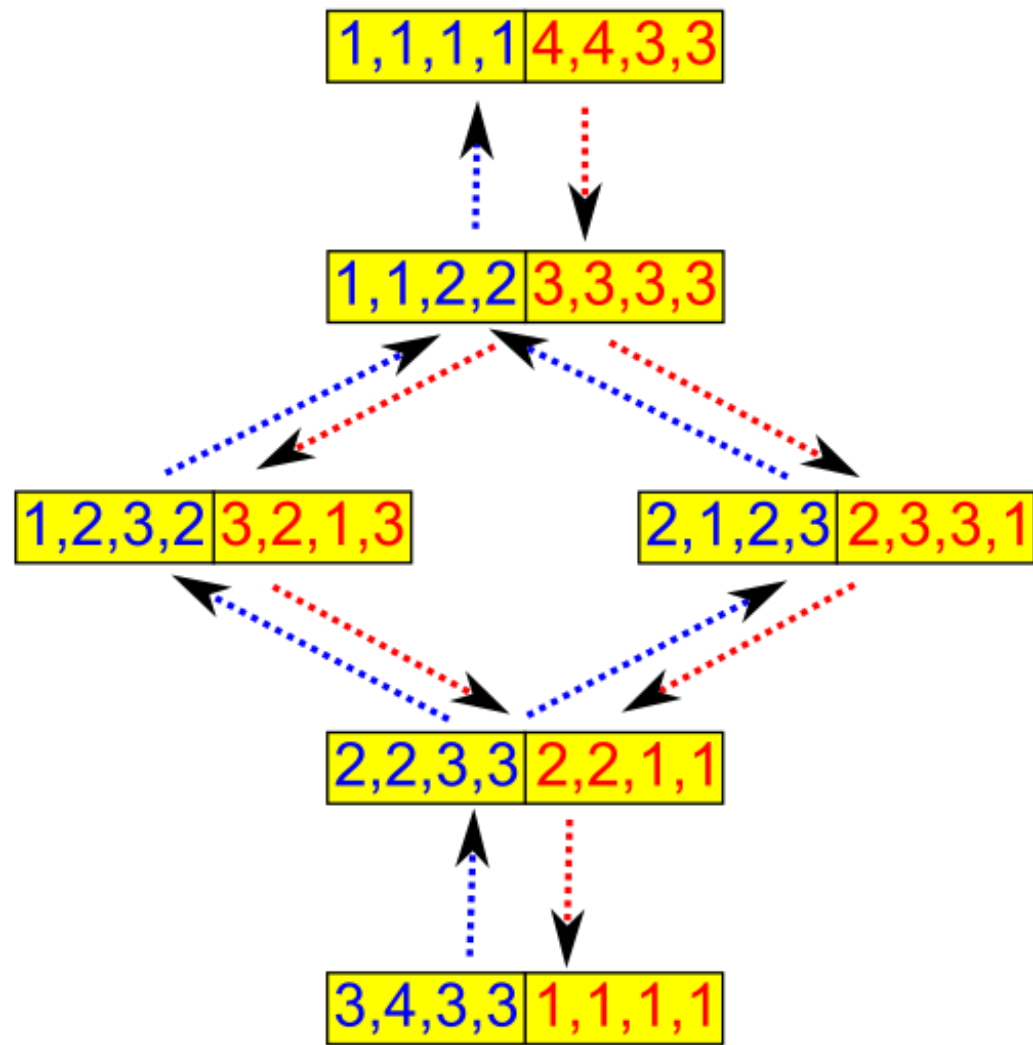
1

2

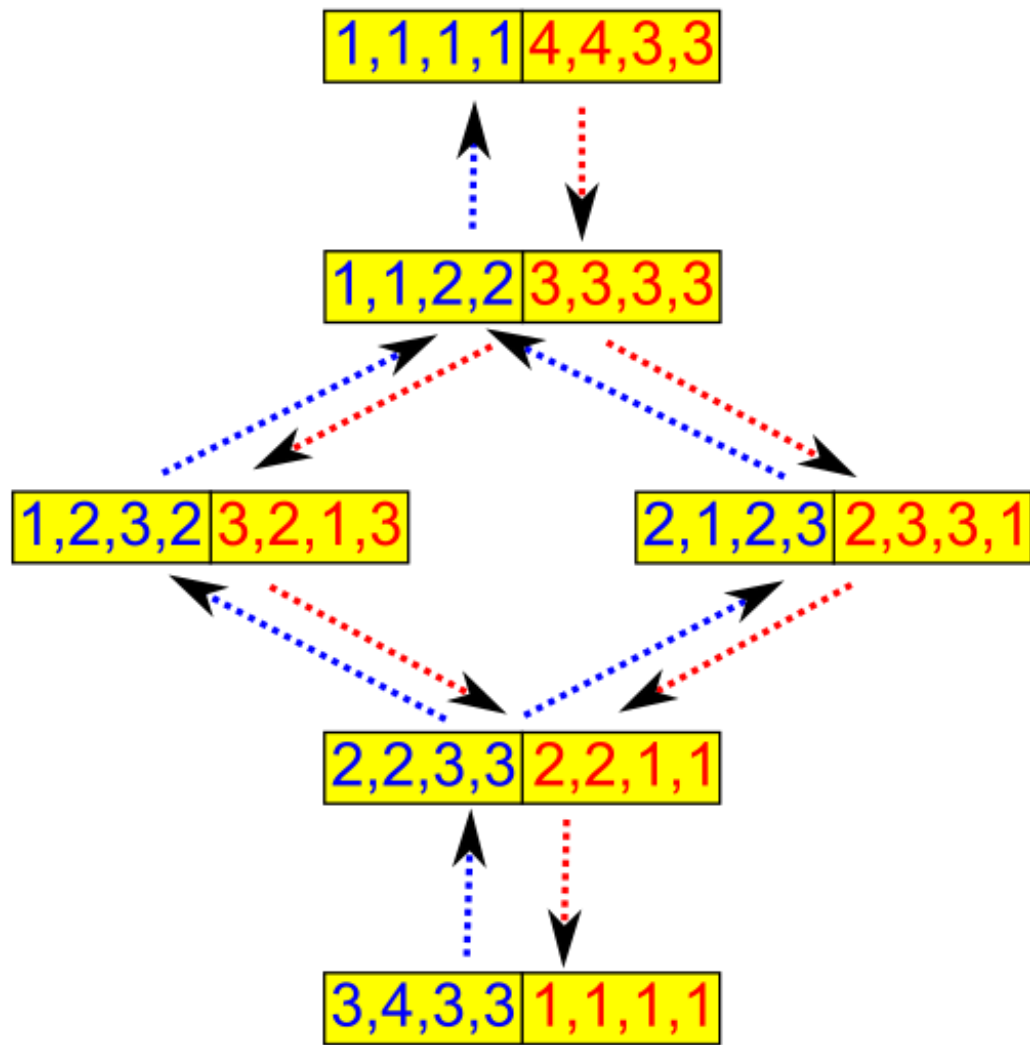
2

3

Consider a uniform combination of all integral stable matchings.

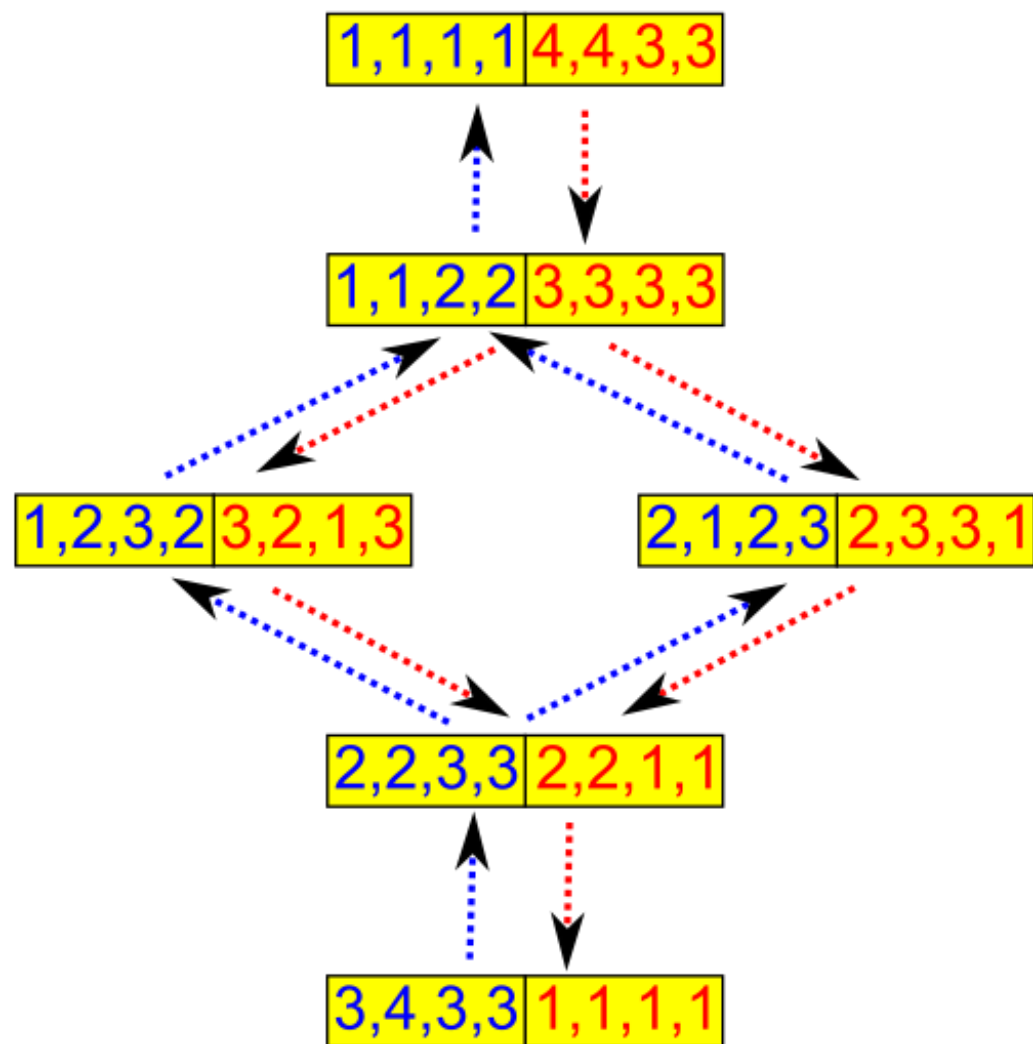


Consider a uniform combination of all integral stable matchings.



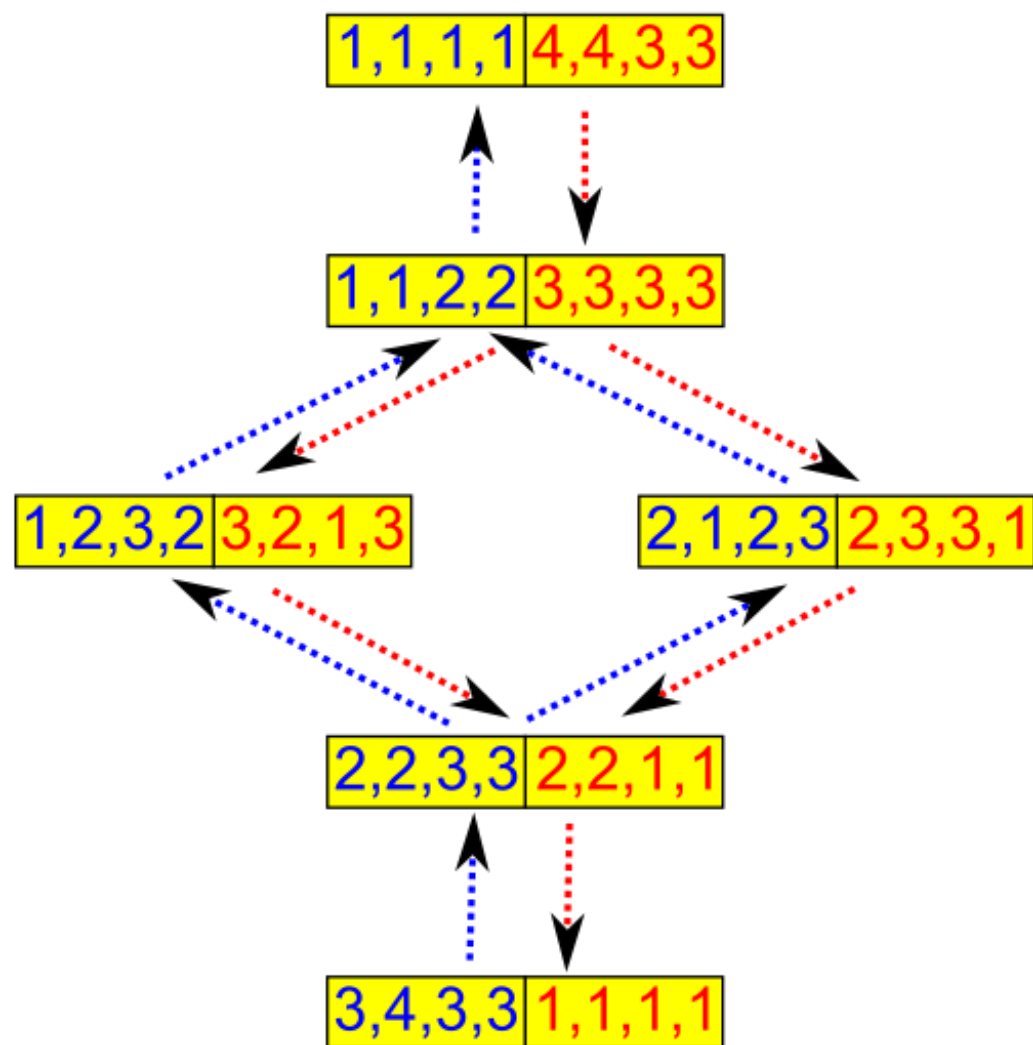
| | 0 | | | | | | 1 |
|-------|---|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 | |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 | |

Consider a uniform combination of all integral stable matchings.



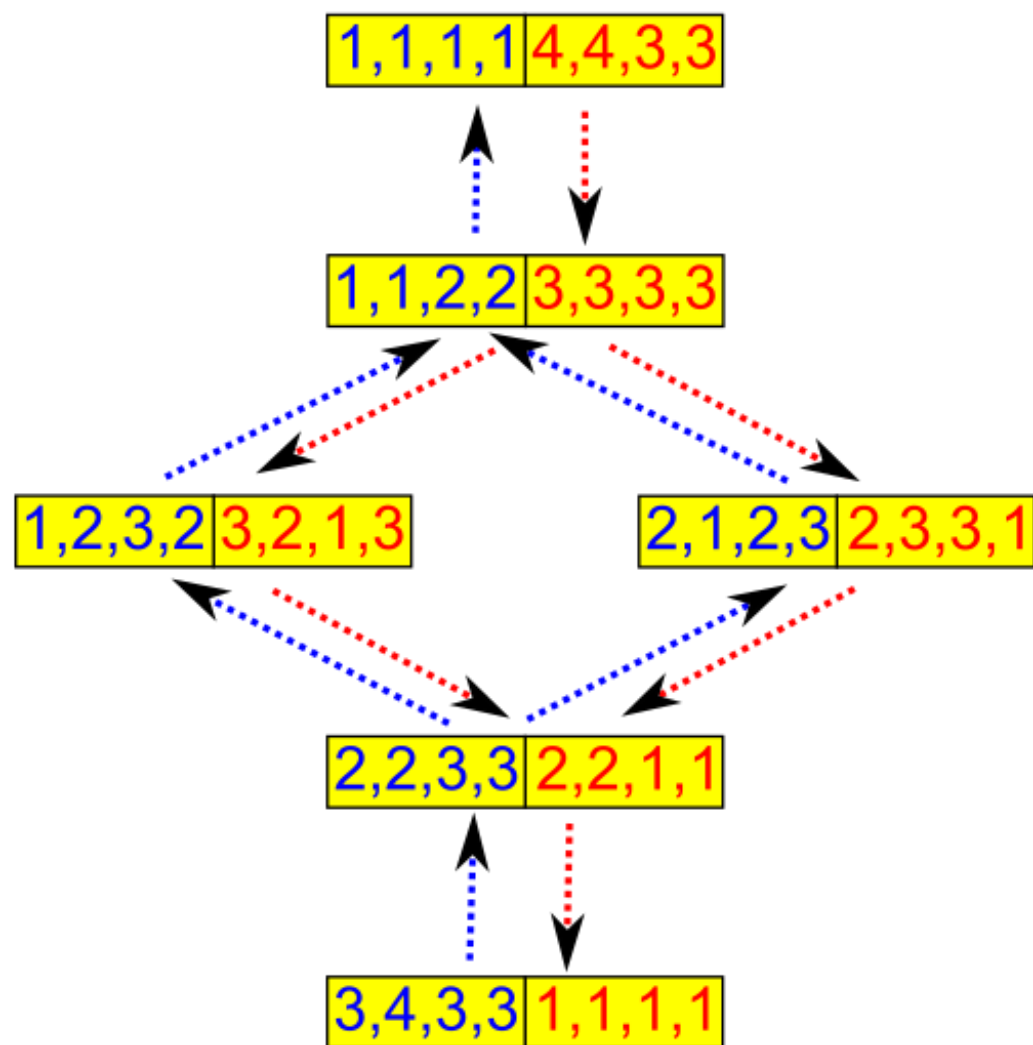
| | 0 | | | | | 1 |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |
| w_1 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_2 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_3 | 3 | 3 | 3 | 1 | 1 | 1 |
| w_4 | 3 | 3 | 3 | 1 | 1 | 1 |

Consider a uniform combination of all integral stable matchings.



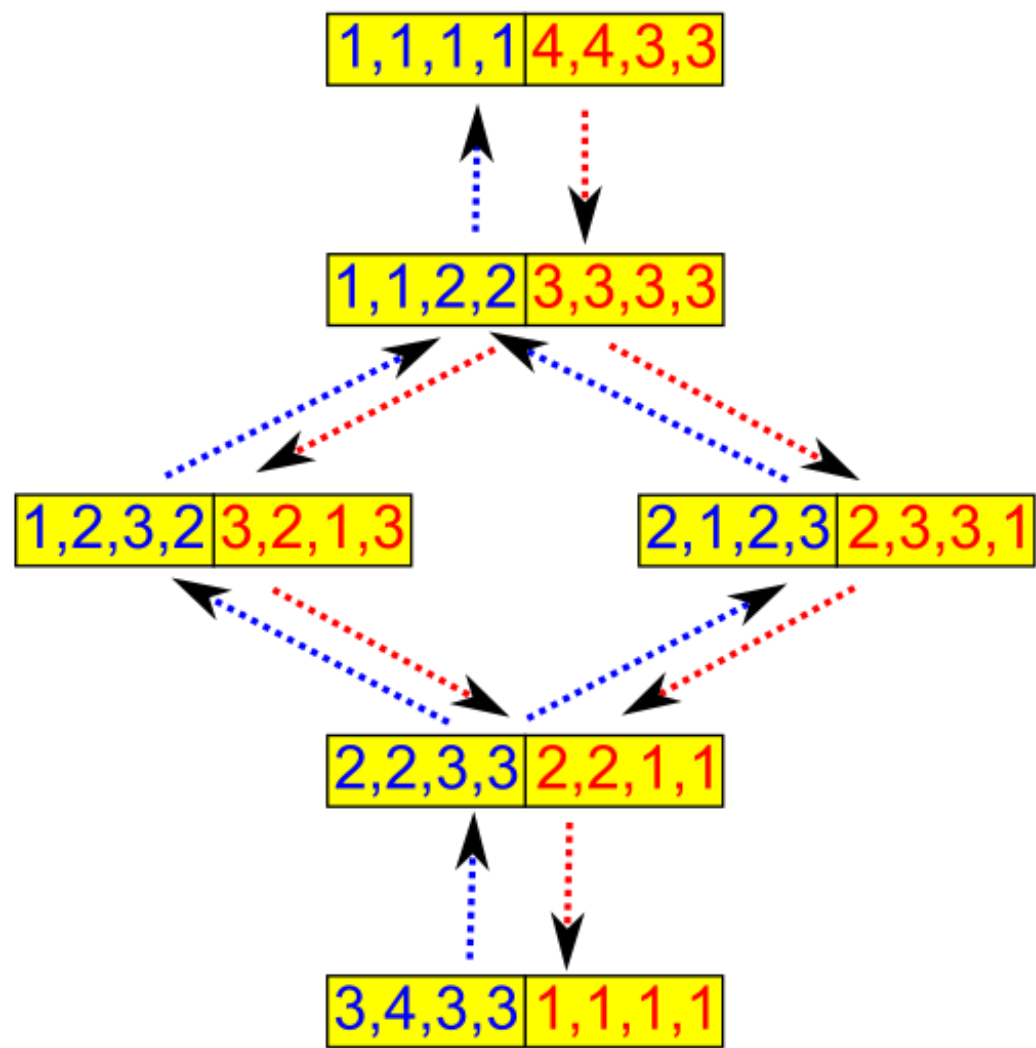
| | 0 | | | | | 1 |
|-------|---|---|---|---|---|---|
| m_1 | 1 | 1 | 1 | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |
| w_1 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_2 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_3 | 3 | 3 | 3 | 1 | 1 | 1 |
| w_4 | 3 | 3 | 3 | 1 | 1 | 1 |

Consider a uniform combination of all integral stable matchings.



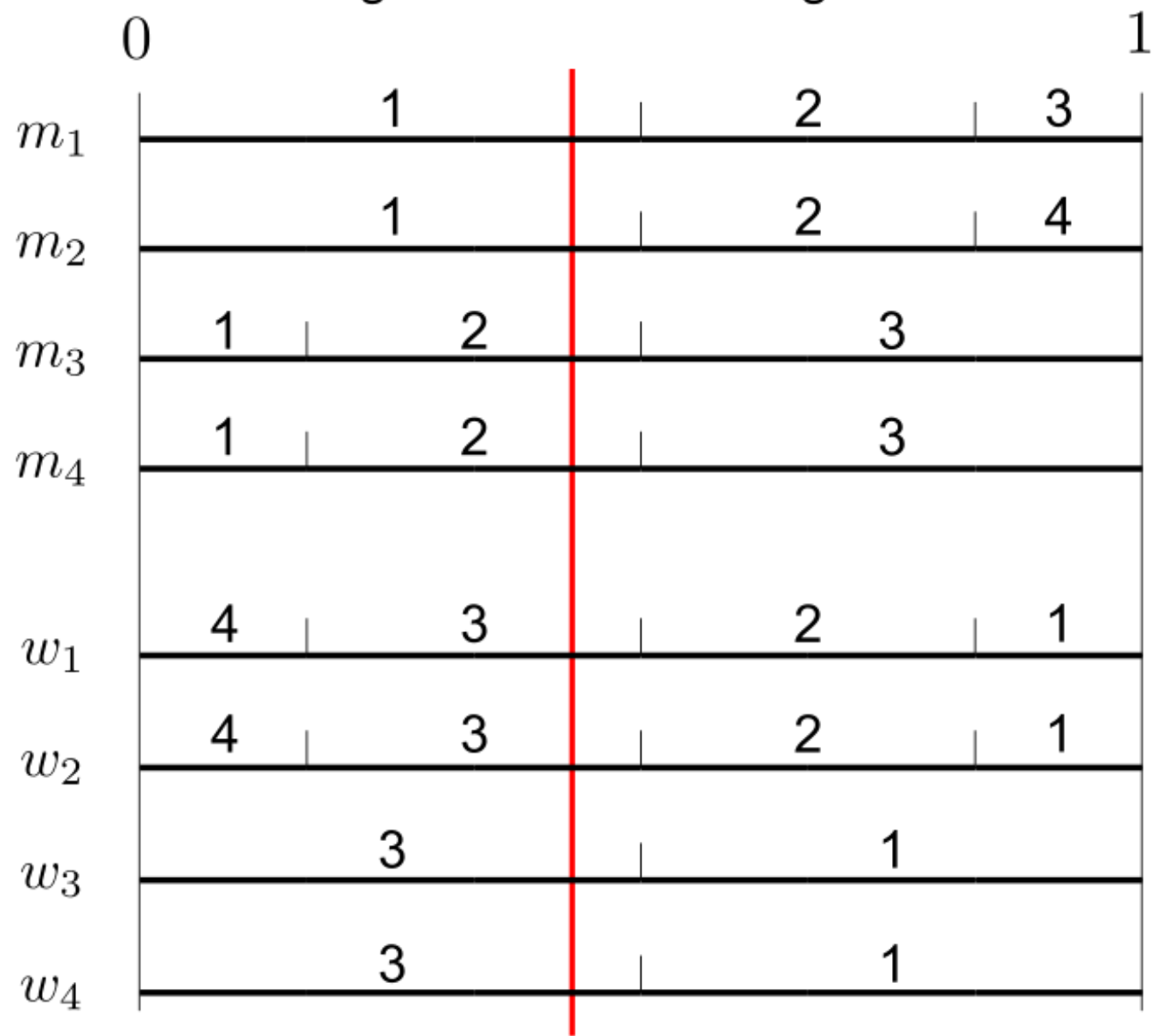
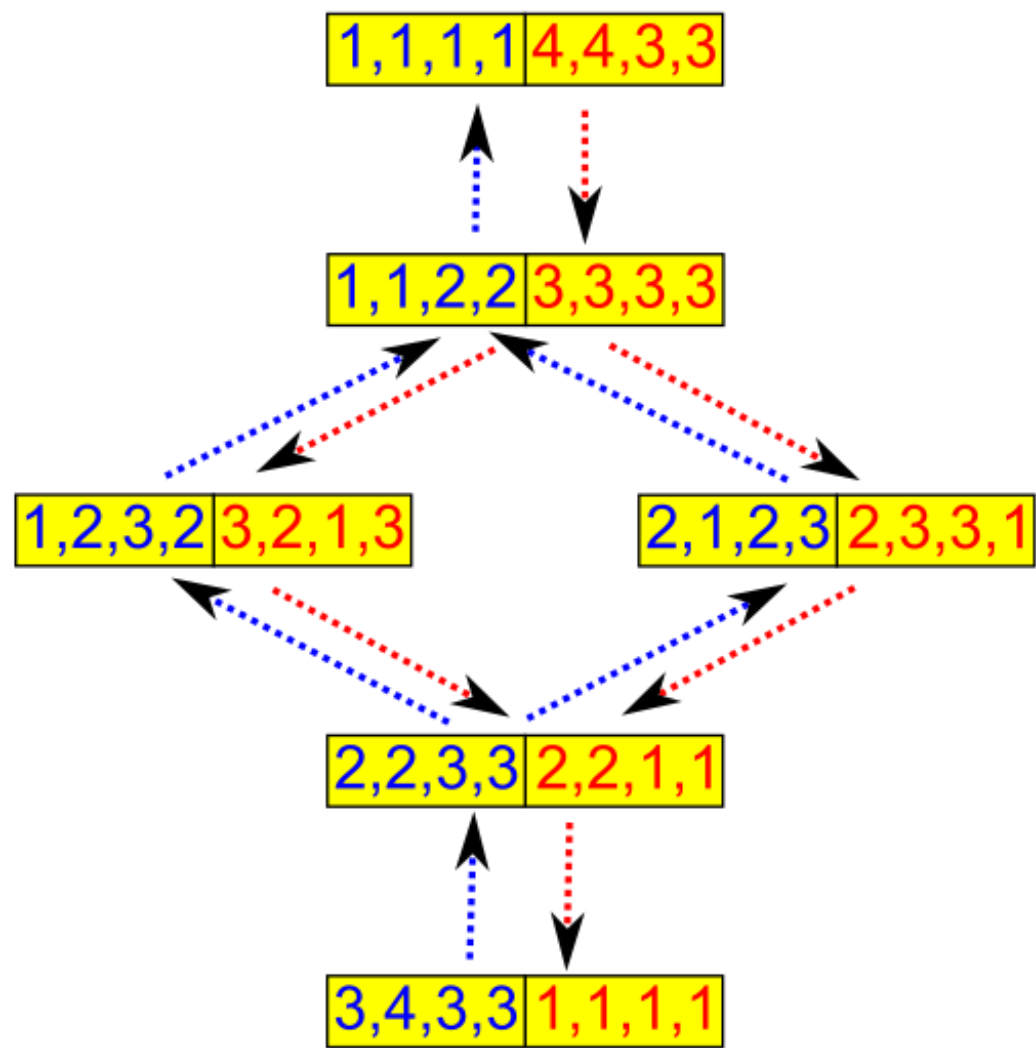
| | 0 | | | | | 1 |
|-------|---|---|---|---|---|---|
| m_1 | | 1 | | 2 | 2 | 3 |
| m_2 | 1 | 1 | 1 | 2 | 2 | 4 |
| m_3 | 1 | 2 | 2 | 3 | 3 | 3 |
| m_4 | 1 | 2 | 2 | 3 | 3 | 3 |
| w_1 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_2 | 4 | 3 | 3 | 2 | 2 | 1 |
| w_3 | 3 | 3 | 3 | 1 | 1 | 1 |
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Consider a uniform combination of all integral stable matchings.

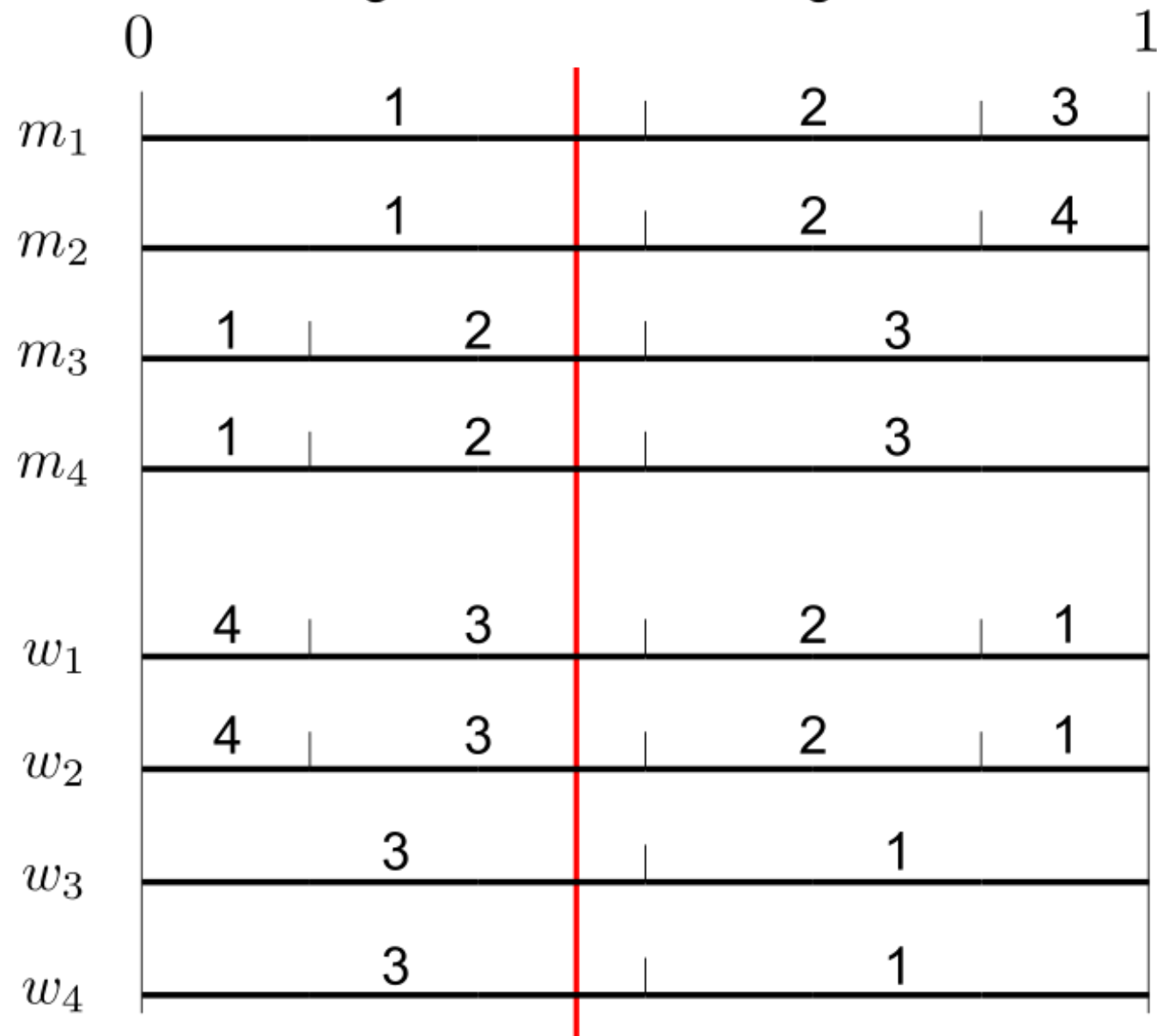
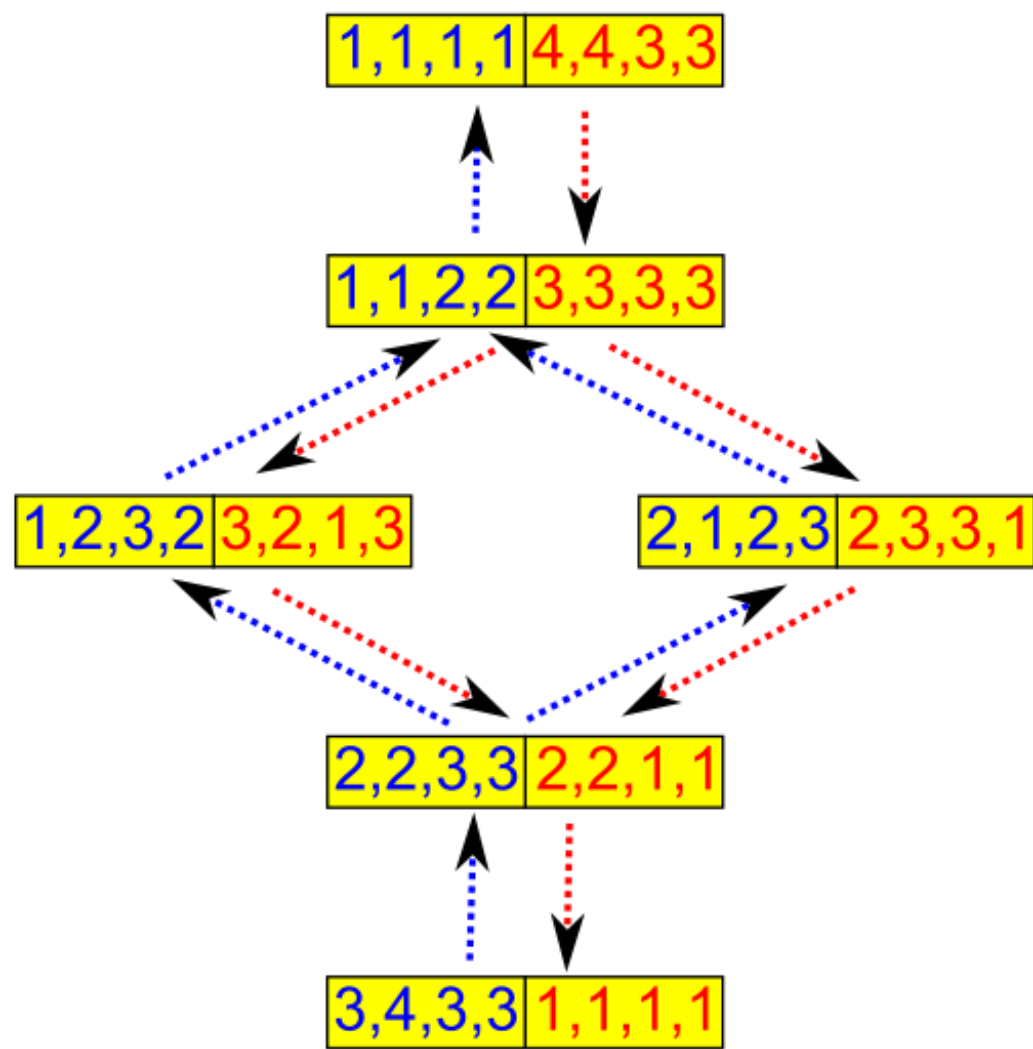


| | 0 | | | | 1 | |
|-------|---|---|---|---|---|---|
| m_1 | | 1 | | 2 | 3 | |
| m_2 | | 1 | | 2 | 4 | |
| m_3 | 1 | | 2 | | 3 | |
| m_4 | 1 | | 2 | | 3 | |
| w_1 | 4 | | 3 | | 2 | 1 |
| w_2 | 4 | | 3 | | 2 | 1 |
| w_3 | | 3 | | | 1 | |
| w_4 | | 3 | | | 1 | |

Consider a uniform combination of all integral stable matchings.

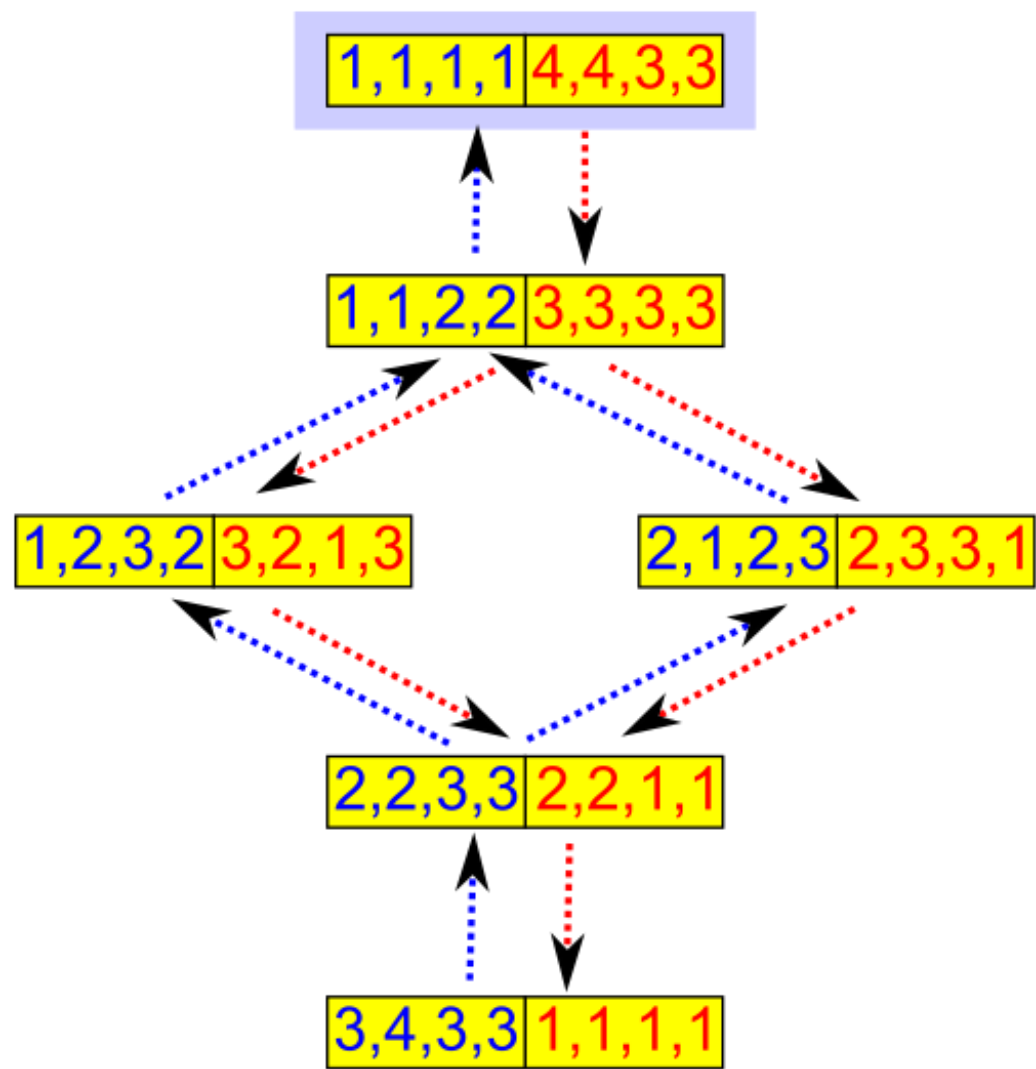


Consider a uniform combination of all integral stable matchings.



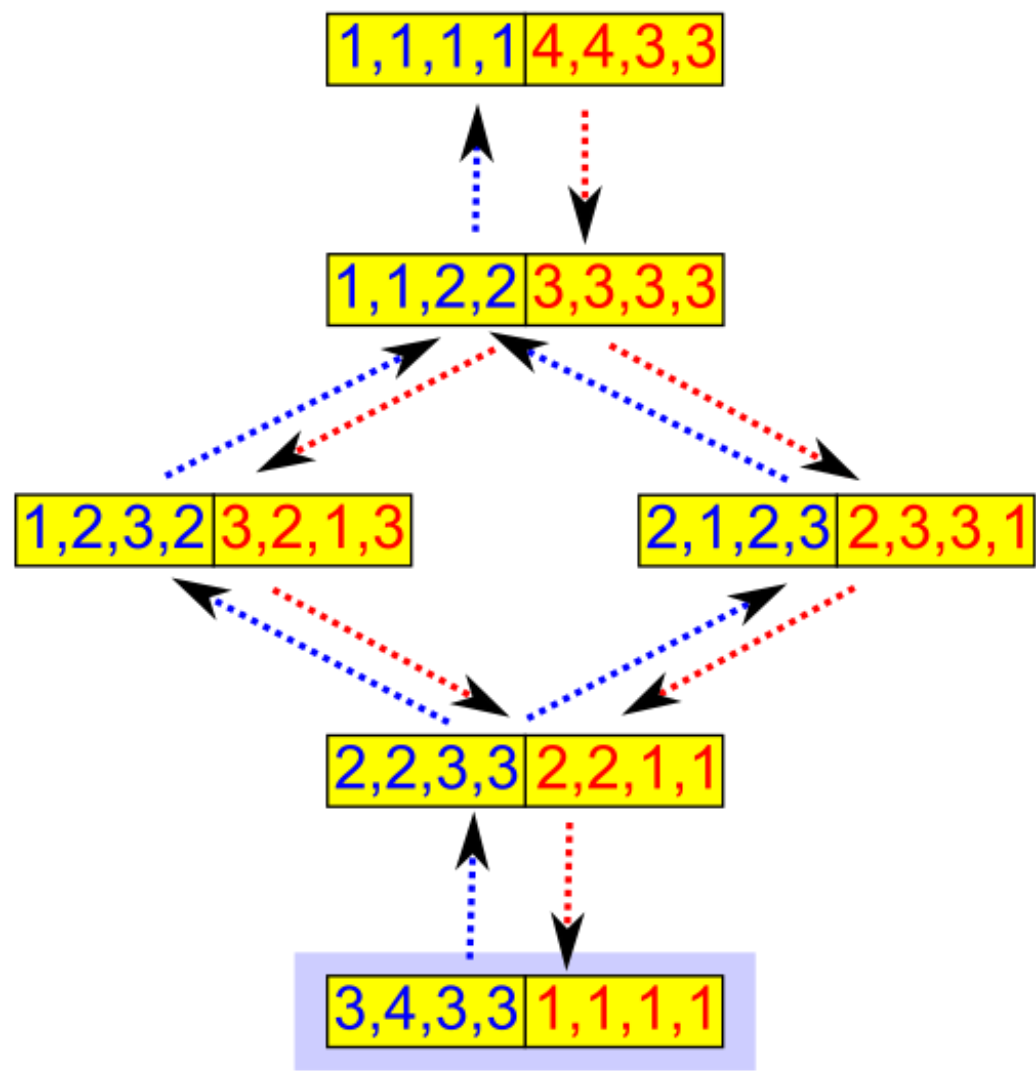
A vertical cut at any point in $[0,1]$ induces a stable matching.

Consider a uniform combination of all integral stable matchings.



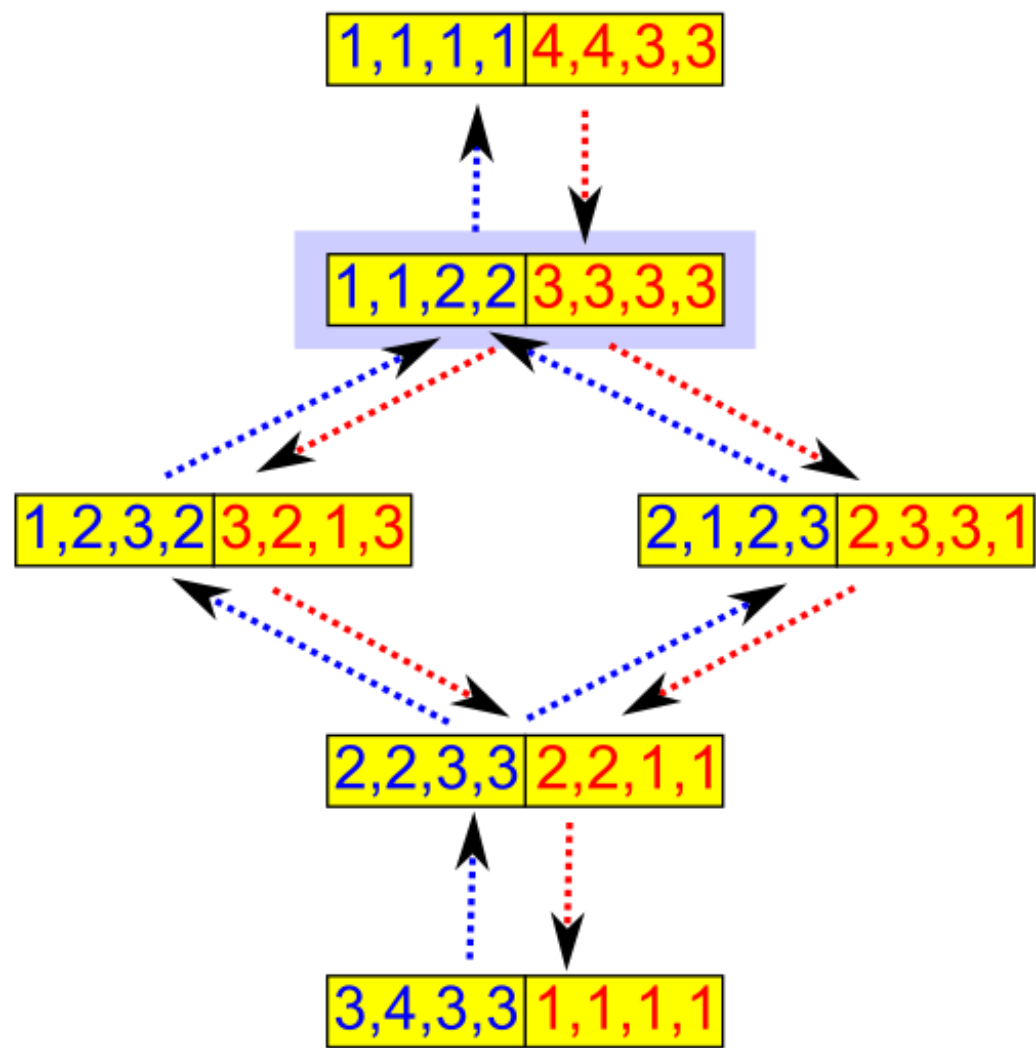
| | 0 | | | | 1 | |
|-------|---|---|---|---|---|---|
| m_1 | | 1 | | 2 | 3 | |
| m_2 | | 1 | | 2 | 4 | |
| m_3 | 1 | | 2 | | 3 | |
| m_4 | 1 | | 2 | | 3 | |
| w_1 | 4 | | 3 | | 2 | 1 |
| w_2 | 4 | | 3 | | 2 | 1 |
| w_3 | | 3 | | | 1 | |
| w_4 | | 3 | | | 1 | |

Consider a uniform combination of all integral stable matchings.



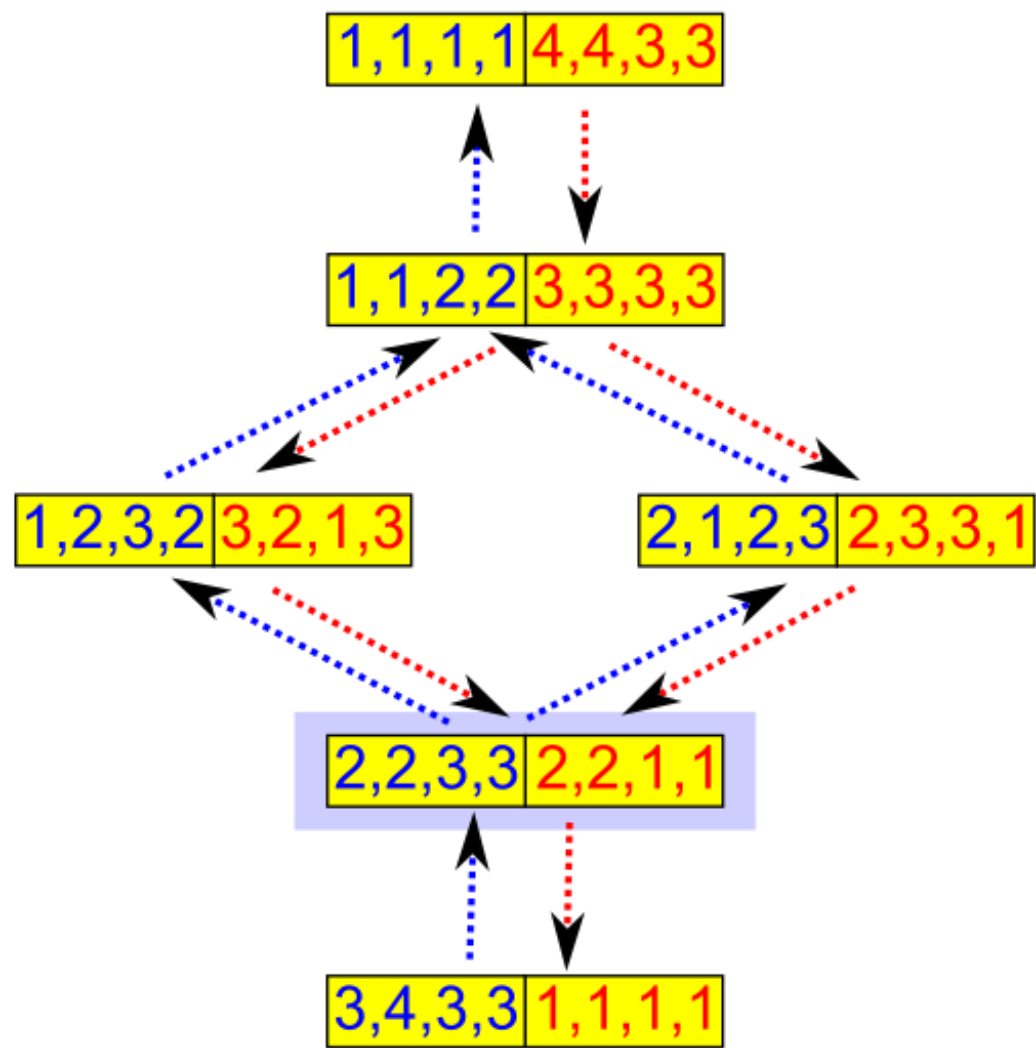
| | 0 | | | | 1 |
|-------|---|---|---|---|---|
| m_1 | | 1 | | 2 | 3 |
| m_2 | | 1 | | 2 | 4 |
| m_3 | 1 | | 2 | | 3 |
| m_4 | 1 | | 2 | | 3 |
| w_1 | 4 | | 3 | | 2 |
| w_2 | 4 | | 3 | | 2 |
| w_3 | | 3 | | | 1 |
| w_4 | | 3 | | | 1 |

Consider a uniform combination of all integral stable matchings.



| | 0 | | 1 |
|-------|---|---|-----|
| m_1 | | 1 | 2 3 |
| m_2 | | 1 | 2 4 |
| m_3 | 1 | 2 | 3 |
| m_4 | 1 | 2 | 3 |
| w_1 | 4 | 3 | 2 1 |
| w_2 | 4 | 3 | 2 1 |
| w_3 | | 3 | 1 |
| w_4 | | 3 | 1 |

Consider a uniform combination of all integral stable matchings.



| | 0 | | 1 |
|-------|---|---|---|
| m_1 | 1 | | 3 |
| m_2 | 1 | | 4 |
| m_3 | 1 | 2 | 3 |
| m_4 | 1 | 2 | 3 |
| w_1 | 4 | 3 | 1 |
| w_2 | 4 | 3 | 1 |
| w_3 | | 3 | 1 |
| w_4 | | 3 | 1 |

Note: A vertical blue shaded region covers the column between 0 and 1 for rows m_1 through w_4 .

Bad News

Bad News

[Cheng, *Algorithmica* 2010]

Computing a median stable matching is **#P-hard**.

Bad News

[Cheng, *Algorithmica* 2010]

Computing a median stable matching is **#P-hard**.

as hard as...

- computing the permanent of a 0-1 matrix
- counting the number of perfect matchings in a bipartite graph
- counting the number of stable matchings for a given instance
- and many others...

Other Notions of Fairness

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Egalitarian

[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the
average rank of matched partners
across all men and women

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[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the *average rank* of matched partners across all men and women

Minimum regret

[Knuth, 1976 (attributed to Selkow)]

a stable matching that minimizes the *maximum rank* of matched partner of any man or woman

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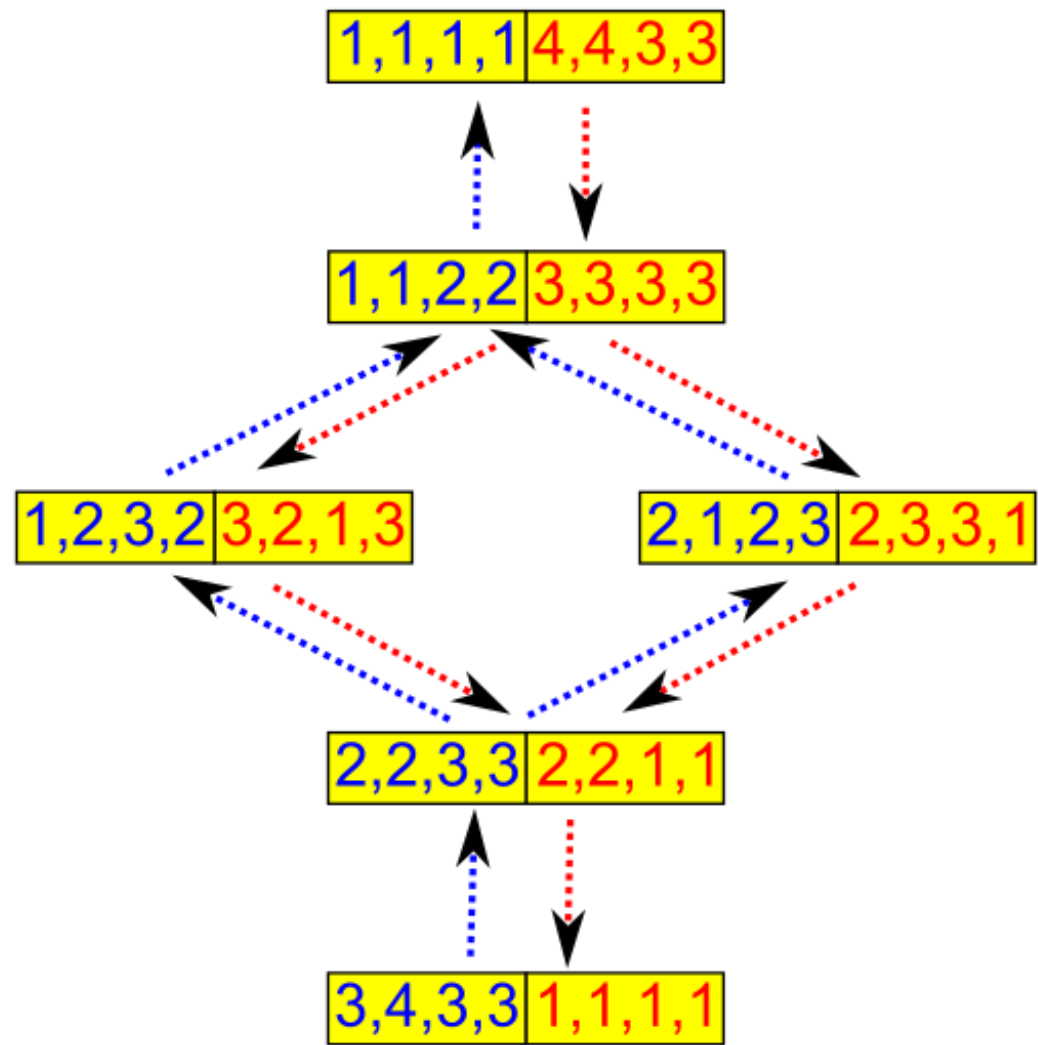
Polynomial time

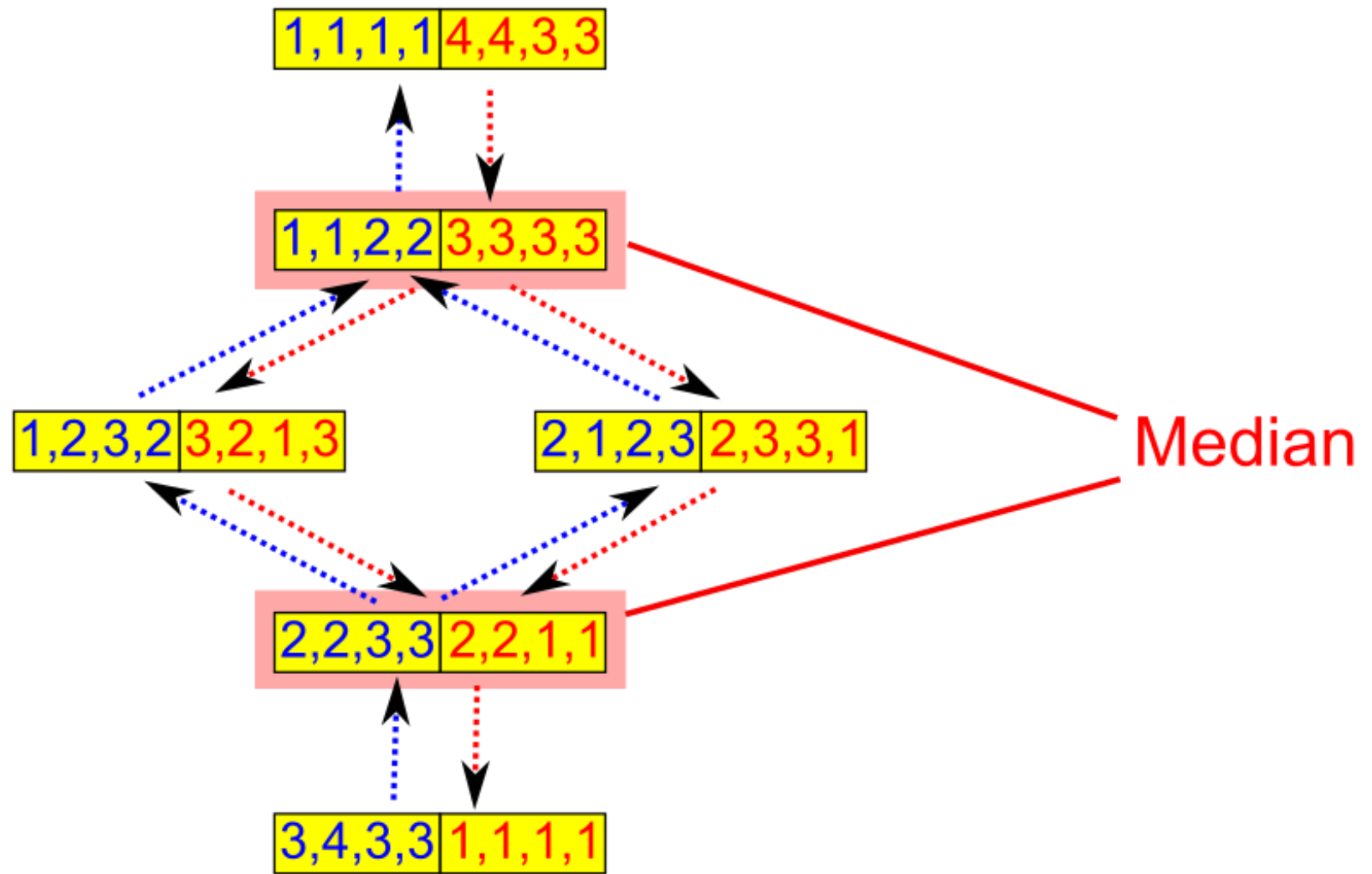
Minimum regret

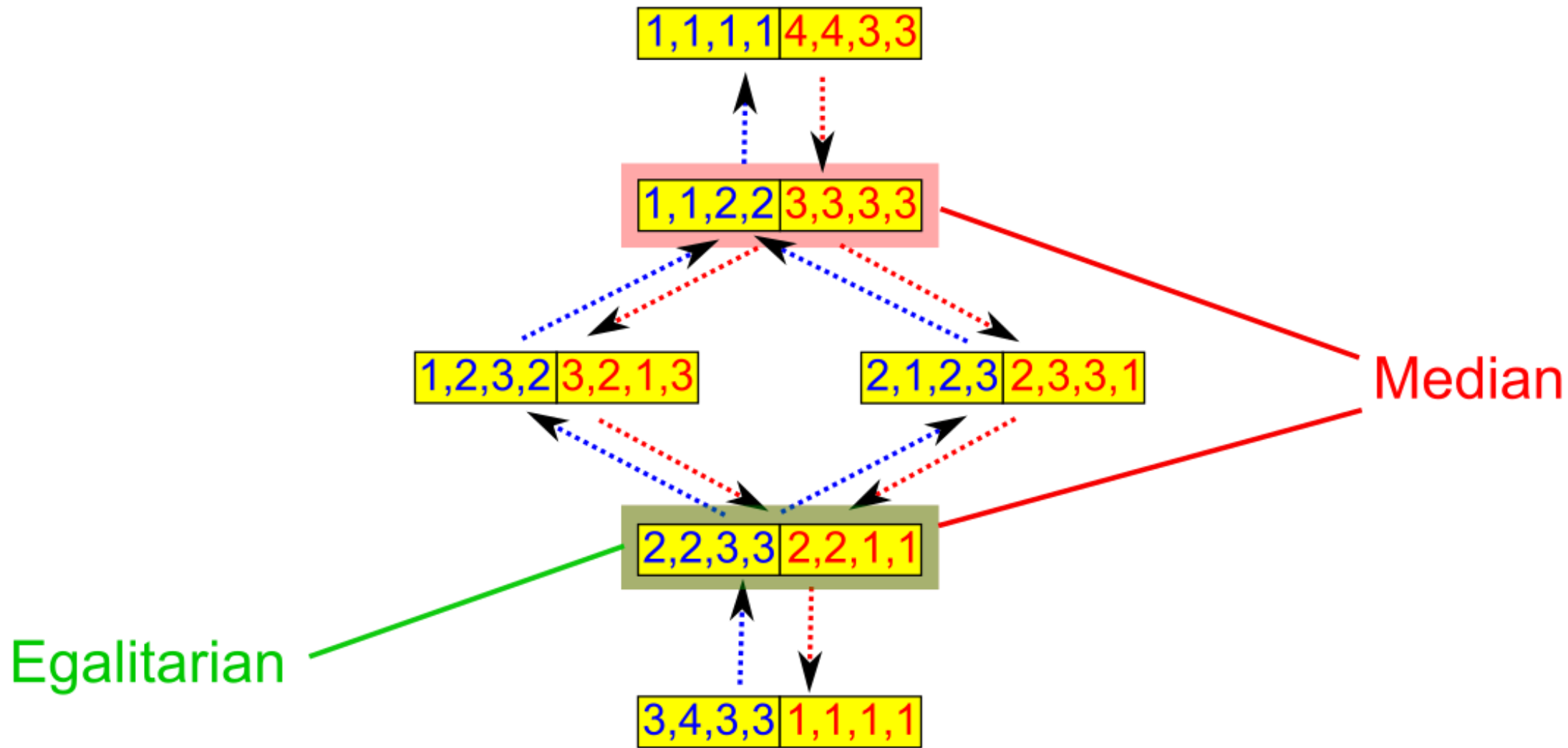
[Knuth, 1976 (attributed to Selkow)]

a stable matching that minimizes the *maximum rank* of matched partner of any man or woman

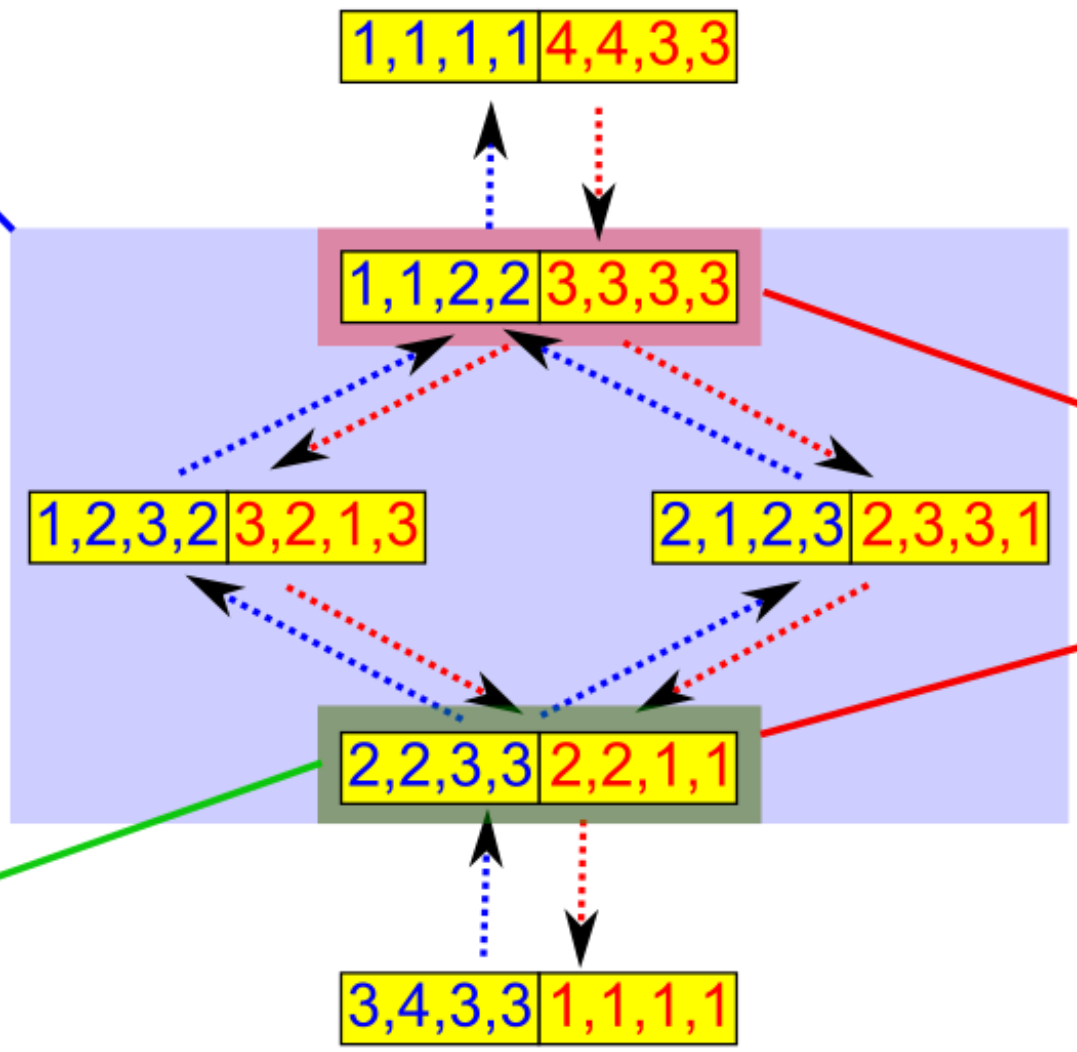
Polynomial time







Minimum Regret



Median

Egalitarian

Summarizing the Matching Landscape

Summarizing the Matching Landscape



One-sided matching

- Housing markets and TTCA
- Kidney exchange

Summarizing the Matching Landscape



One-sided matching

- Housing markets and TTCA
- Kidney exchange



Two-sided matching

- Stable matchings and DA algorithm
- Lattice of stable matchings

Summarizing the Matching Landscape



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Incentives

- Strategyproofness vs stability
- One-sided strategyproofness of DA
- Optimal strategy is "inconspicuous"

Summarizing the Matching Landscape



One-sided matching

- Housing markets and TTCA
- Kidney exchange



Two-sided matching

- Stable matchings and DA algorithm
- Lattice of stable matchings



Fairness

Median stable matching
always exists.



Incentives

- Strategyproofness vs stability
- One-sided strategyproofness of DA
- Optimal strategy is "inconspicuous"

Reminders

Course feedback on Moodle (by March 10)

Sign up for project progress meeting

Next Time

Cake cutting



References

- Slides by Christine Cheng on “Fair Stable Matchings”.
<https://www.optimalmatching.com/MATCHUP2015/slides/ChristineCheng.pdf>
- Linear programming-based formulation of the stable matching problem.

John Vande Vate

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- A median stable matching always exists.

Chung-Piaw Teo and Jay Sethuraman

“The Geometry of Fractional Stable Matchings and Its Applications”

MOR, 23(4), 1998 pg 874-891

<https://pubsonline.informs.org/doi/abs/10.1287/moor.23.4.874>

- Computing a median stable matching is #P-hard.

Christine Cheng

“Understanding the Generalized Median Stable Matchings”

Algorithmica, 58, 2010 pg 34-51

<https://link.springer.com/article/10.1007/s00453-009-9307-2>

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Robert Irving, Paul Leather, and Dan Gusfield

“An Efficient Algorithm for the Optimal Stable Marriage Problem”

JACM, 34, 1987 pg 532-543

<https://dl.acm.org/doi/pdf/10.1145/28869.28871>

